Innovations in Incidence Geometry
Volume 14 (2015), Pages 81–110
ISSN 1781-6475

An elementary description of the Mathieu dual hyperoval and its splitness

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Abstract

An elementary new construction of a 3-dimensional dual hyperoval $\mathcal{M}$ over $\mathbb{F}_4$ is given, as well as an explicit analysis of the structure of its automorphism group. This provides a self-contained introduction to the Mathieu simple group M_{22}. The basic properties of $\mathcal{M}$ as a dimensional dual hyperoval, e.g. splitness, complements, linear systems, quotients and coverings, are derived from this construction.

Keywords: dimensional dual hyperoval (DHO), the Mathieu DHO, Mathieu group M_{22}, automorphism group, complement, linear system.
MSC 2010: 20B25, 20D08, 05B25, 05E18, 51A45, 51E20

1 Introduction

The purpose of this paper is to derive basic properties such as the automorphism group, complements and related DHO-sets, substructures, coverings and quotients, of a geometric structure $\mathcal{M}$, called the Mathieu dual hyperoval [9], based on its new simple construction. Specifically, this contains an elementary and self-contained introduction to the Mathieu simple group M_{22} of degree 22, because M_{22} appears as a quotient group of the linear part of the automorphism group of $\mathcal{M}$.

As far as the author knows, the existence of $\mathcal{M}$ was first observed by Jónsson and McKay in terms of the Leech lattice [4, Claim (2.13)], which was explicitly given in a table form using the MOG arrangement [1]. Nakagawa gave an account how to obtain this table, assuming that there exists a certain unitary dual hyperoval [6]. Several characterizations of this structure as dual hyperovals are available (e.g. [2, 5, 6]).

Now let me give more details on the results. A collection $\mathcal{S}$ of $1 + \sum_{i=0}^{n-1} q^i$ subspaces of dimension n in a vector space U over $\mathbb{F}_q$ is called a *dimensional dual hyperoval* (abbreviated to DHO) of rank n over $\mathbb{F}_q$, if any two distinct members of them intersect at a 1-dimensional subspace, but any three mutually distinct members intersect at the zero space. The subspace of U spanned by all members of $\mathcal{S}$ is called the *ambient space* of $\mathcal{S}$. If there exists a subspace Y of codimension n in the ambient space of $\mathcal{S}$ which intersects every member of $\mathcal{S}$ at the zero space, we say that $\mathcal{S}$ *splits over* Y. Such a subspace is called a *complement* to $\mathcal{S}$. We say that $\mathcal{S}$ is *of split type*, if there is a complement to $\mathcal{S}$. For a complement Y to $\mathcal{S}$ and a member X of $\mathcal{S}$, we can associate a collection $\mathcal{L}_X(\mathcal{S}; Y)$ of linear maps from X to Y, which we call the *DHO*-set following [3]. For the details, see Section 3.2. The above-mentioned structure $\mathcal{M}$ is a DHO of rank 3 over $\mathbb{F}_4$ with ambient space the 6-dimensional unitary space U. It is *unitary*, in the sense that all members of $\mathcal{M}$ are totally isotropic with respect to the unitary form on U. In Section 2 of this paper, regarding U as the symmetric square tensor product $S^2(V)$ of a 3-dimensional space V, the DHO $\mathcal{M}$ is constructed as a deformation of the standard DHO, the Veronesean DHO $\mathcal{V}_3(\mathbb{F}_4)$ (see, e.g., [7]), with extra diagonal terms given by a map ι on V. The map ι is quadratic having the exterior product $\times$ as the associated alternating bilinear map (see Section 2.1).

Based on this description, the linear automorphism group $L(\mathcal{M})$ of $\mathcal{M}$ is determined in a self-contained and explicit manner (see Section 2.3). Specifically, the stabilizer in $L(\mathcal{M})$ of a member A of $\mathcal{M}$ are described as a group of explicit unitary matrices of degree 6 (see Proposition 2.5). The factor group of $L(\mathcal{M})$ by its center is shown to be a non-abelian simple group of order $2^7.3^2.5.7.11$ (in fact, isomorphic to the Mathieu group M_{22} as we shall see in Section 2.3.1). Thus Section 2 can be regarded as a new and self-contained introduction to the Mathieu simple group M_{22} in terms of a DHO.

In Section 3 of this paper, the splitness of $\mathcal{M}$ is established, using a characterization of a complement in terms of some alternating maps (see Section 3.1). Then an explicit linear system is calculated over a complement, which provides a collection of unitary matrices with certain properties (see Section 3.3). Finally, it is shown that there are exactly $2^5.5.7.11$ complements to $\mathcal{M}$, which are all non-degenerate with respect to the unitary form on U and form a single orbit under the action of $L(\mathcal{M})$ (see Section 3.4). In general, it is not straightforward to determine all complements to a DHO, and the complements are split into more than one orbits under the automorphism group of the DHO in question (see, for example, [10]). Exploiting explicit automorphisms obtained in Section 2.3, this can be done for the DHO $\mathcal{M}$ by elementary group theoretic arguments with straightforward calculations by hand. The paper is concluded with remarks on substructures, coverings and quotients of $\mathcal{M}$ (see Section 4).

2 A simple model of the Mathieu DHO

2.1 Notation and review about the exterior product

In this section, we shall use the letters i, j and k to denote indices in $\{0,1,2\}$ unless otherwise stated. Given an index i in $\{0,1,2\}$, we adopt the convention that

the letters j and k denote the remaining indices, so that $\{i,j,k\}=\{0,1,2\}$.

For example, $\sum_{i=0}^{2} x_j x_k e_i$ denotes $x_1x_2e_0+x_2x_0e_1+x_0x_1e_2$, and so on.

Let V be a vector space of dimension 3 over $\mathbb{F}_4$ with basis e_0, e_1, e_2. For a vector $x\in V$ and an index i in $\{0,1,2\}$, the letter x_i denotes the coefficient of e_i in the expression $x=x_0e_0+x_1e_1+x_2e_2$ (x_0, x_1, $x_2\in\mathbb{F}_4$). We define a nondegenerate symmetric bilinear form on V, the *dot product*, by $x\cdot y:=\sum_{i=0}^{2}x_iy_i$. For $\alpha\in\mathbb{F}_4$, we set $\overline{\alpha}:=\alpha^2$. Throughout the paper, ω denotes a primitive element of $\mathbb{F}_4$.

We denote by $S^2(V)$ the square symmetric tensor product of V, i.e., the factor space $(V\otimes V)/W$ of $V\otimes V$ by the subspace W spanned by $x\otimes y+y\otimes x$ for all $x,y\in V$. The element $(x\otimes y)+W$ in $S^2(V)$ with $x,y\in V$ is denoted by $x\otimes y$ for short. (This notation should not cause any confusion, because we only work with $S^2(V)$ later.) For $x\in V$, we use the symbol $\Delta(x)$ to denote a vector $x\otimes x$ in $S^2(V)$. Notice that $\Delta(x+y)=\Delta(x)+\Delta(y)$ and $\Delta(\alpha x)=\overline{\alpha}\Delta(x)$ for $x,y\in V$, $\alpha\in\mathbb{F}_4$. We adopt the following vectors as a basis for $S^2(V)$:

$$\Delta_i:=\Delta(e_i) \text{ and } \nabla_i:=e_j\otimes e_k \text{ for } i\in\{0,1,2\},$$

where we follow the convention above. Explicitly, $\nabla_0:=e_1\otimes e_2$, $\nabla_1:=e_2\otimes e_0$ and $\nabla_2:=e_0\otimes e_1$. For a vector $v=\sum_{i=0}^{2}v_i\Delta_i+\sum_{i=0}^{2}v_i'\nabla_i$ in $S^2(V)$, the terms $v_i\Delta_i$ and $v_i'\nabla_i$ ($i=0,1,2$) are respectively called *diagonal* and *off-diagonal components* of v.

We define a non-degenerate unitary form $(\ ,\ )$ on $S^2(V)$ as follows: define on the basis $(\Delta_i,\Delta_j)=(\nabla_i,\nabla_j)=0$ for all $i,j\in\{0,1,2\}$, and $(\Delta_i,\nabla_j)=1$ or 0 according as $i=j$ or not, and extend it semi-linearly on $S^2(V)$.

Now we introduce a map ι on V which is so-called *quadratic*, in the sense that the associated map sending $(x,y)\in V\times V$ to $\iota(x+y)+\iota(x)+\iota(y)$ $(\in V)$ is an (alternating) bi-additive map (in fact, bilinear over $\mathbb{F}_4$). For $x=\sum_{i=0}^{2}x_ie_i\in V$, we define

$$\iota(x):=\sum_{i=0}^{2}x_jx_ke_i=x_1x_2e_0+x_2x_0e_1+x_0x_1e_2, \tag{1}$$

where the indices i, j, k follow our convention above. Notice that $\iota(e_i) = 0$ and $\iota(x_i e_i + x_j e_j) = x_i x_j e_k$. The associated map $\iota(x+y)+\iota(x)+\iota(y)$ is calculated as $\sum_i((x+y)_j(x+y)_k + x_j x_k + y_j y_k)e_i = \sum_i(x_j y_k + x_k y_j)e_i$. This is an alternating bilinear map on V, called the *exterior product* $x \times y$ of x and y:

$$x \times y := \sum_{i=0}^{2}(x_j y_k + x_k y_j)e_i, \tag{2}$$

with our convention $\{i, j, k\} = \{0, 1, 2\}$ on indices. Hence we have

$$\iota(x + y) + \iota(x) + \iota(y) = x \times y \tag{3}$$

for any $x, y \in V$. This relation and the following basic properties of the exterior product are frequently used in the sequel. We have $x \times x = 0$, $x \times y = y \times x$ and $(x + y) \times z = (x \times z) + (y \times z)$ for $x, y, z \in V$. Furthermore, $x \times y = 0$ if and only if x and y are linearly dependent. Given explicit vectors x and y in V, $x \times y$ is easily obtained by using the bilinearity and the fact that $e_i \times e_j = e_k$ and $e_i \times e_i = 0$.

We give two fundamental equations on the exterior and the dot products in V, which are easy to verify: for vectors x, y, z in V,

$$(x \times y) \cdot z = z \cdot (x \times y) = \det(r(x, y, z)), \tag{4}$$

$$x \times (y \times z) = (x \cdot z)y + (x \cdot y)z, \tag{5}$$

where $r(x, y, z)$ denotes the square matrix of degree 3 with ${}^t(x_j, y_j, z_j)$ as its j-th column for expressions $x = \sum_j x_j e_j$, $y = \sum_j y_j e_j$, $z = \sum_j z_j e_j$. In the sequel, we shall often use the following corollary to equation (4):

> For all $x, y \in V$, we have $x \cdot (x \times y) = y \cdot (x \times y) = 0$. In particular, if x and y are linearly independent, they form a basis for the 2-dimensional subspace $\{z \in V \mid z \cdot (x \times y) = 0\} = (x \times y)^{\perp}$ of V.

Furthermore, the following equations for $x, y, z \in V$ will be used later (notice $\{i, j, k\} = \{0, 1, 2\}$ in equation (6) by our convention):

$$(x \times y)_j(x \times z)_k + (x \times y)_k(x \times z)_j = x_i \det(r(x, y, z)), \tag{6}$$

$$(x \times y)_0(x \times y)_1(x \times y)_2 = \sum_{i=0}^{2} x_i y_i \overline{(x \times y)_i}. \tag{7}$$

Notice that equation (6) is equivalent to $(x \times y) \times (x \times z) = \delta x$ with $\delta := \det(r(x, y, z))$. Equation (5), applied to $x \times y$, x and z respectively for x, y, z, reads $(x \times y) \times (x \times z) = ((x \times y) \cdot z)x + ((x \times y) \cdot x)z$, which is δx by equation

(4) and its corollary. This verifies equation (6). As for equation (7), notice that the left hand side of equation (7) is $(x \times y) \cdot \iota(x \times y)$, which is equal to $\det(r(x, y, \iota(x \times y)))$ and therefore to $x \cdot (y \times \iota(x \times y))$ as well by equation (4). Now for each $i \in \{0,1,2\}$, we have $(y \times \iota(x \times y))_i = y_j \iota(x \times y)_k + y_k \iota(x \times y)_j = y_j(x\times y)_j(x\times y)_i + y_k(x\times y)_i(x\times y)_k = \{y_j(x_i y_k + x_k y_i) + y_k(x_i y_j + x_j y_i)\}(x\times y)_i = \{\ y_i(x \times y)_i\ \}(x \times y)_i = y_i\overline{(x \times y)_i}$. Thus $x \cdot (y \times \iota(x \times y)) = \sum_{i=0}^{2} x_i y_i \overline{(x \times y)_i}$, which verifies equation (7).

2.2 Construction

For vectors $x = \sum_{i=0}^{2} x_i e_i$ and $y = \sum_{i=0}^{2} y_i e_i$ in V ($x_i, y_i \in \mathbb{F}_4$), we define a vector $m(x, y)$ in $S^2(V)$ by

$$m(x, y) := x \otimes y + \Delta(\iota(x \times y)). \tag{8}$$

Recalling the definition of ι (see equation (1)), the off-diagonal and diagonal components of $m(x, y)$ are $(x \times y)_i \nabla_i$ and $\left(x_i y_i + \overline{(x \times y)_j (x \times y)_k}\right) \Delta_i$ respectively for $i \in \{0,1,2\}$, where the indices j, k are chosen to satisfy $\{i, j, k\} = \{0,1,2\}$ by our convention:

$$m(x, y) = \sum_{i=0}^{2} (x \times y)_i \nabla_i + \sum_{i=0}^{2} \left(x_i y_i + \overline{(x \times y)_j (x \times y)_k}\right) \Delta_i \tag{9}$$

Clearly $m(x, y) = m(y, x)$. For $\alpha \in \mathbb{F}_4$, we have $(\alpha x \times y)_j (\alpha x \times y)_k = \overline{\alpha}(x \times y)_j (x \times y)_k$ and then $\Delta((\alpha x \times y)_j (\alpha x \times y)_k e_i) = \alpha \Delta((x \times y)_j (x \times y)_k e_i)$ (To obtain this equality, we need to work with the four element field.) Thus $m(\alpha x, y) = \alpha m(x, y) = m(x, \alpha y)$. For a nonzero vector v in V, this implies that the subset $\{m(x, v) \mid x \in V\}$ of $S^2(V)$ is determined by the projective point $[v] := \mathbb{F}_4 v$ in $\mathbf{PG}(V)$ through v. We denote

$$A[v] := \{m(x, v) \mid x \in V\}. \tag{10}$$

Furthermore, we set $A := \{\Delta(x) \mid x \in V\}$.

Lemma 2.1. *For $a, b, v \in V$, the following hold with $\delta := \det(r(a, b, v))$*

$$m(a + b, v) + m(a, v) + m(b, v) = \Delta(\delta v), \tag{11}$$

$$m(a, v) + m(b, v) = m(a + b + \overline{\delta} v, v). \tag{12}$$

Proof. From equation (8), $m(0, x) = 0$ and $m(x, x) = \Delta(x)$ for $x \in V$. In view of (8) and the bilinearity of the map sending $(x, y) \in V \times V$ to $x \times y \in S^2(V)$, equation (11) is equivalent to the following equation:

$$\iota((a + b) \times v) + \iota(a \times v) + \iota(b \times v) = \delta v$$

with $\delta := \det(r(a,b,v))$. However, as $(a+b)\times v = (a\times v)+(b\times v)$, it follows from equation (3) that the left hand side of this equation is $(a\times v)\times(b\times v)$, which is equal to $\{(a\times v)\cdot b\}v + \{(a\times v)\cdot v\}b = \delta v$ by the basic equations (4) and (5).

In particular, for $b = \alpha v$ with $\alpha \in \mathbb{F}_4$, we have $\det(r(a,\alpha v,v)) = 0$ and therefore $m(a+\alpha v,v)+m(a,v)+m(\alpha v,v) = 0$, or equivalently

$$m(a+\alpha v,v) = m(a,v)+\alpha\Delta(v) = m(a,v)+\Delta(\overline{\alpha}v).$$

Thus we have $m(a,v)+m(b,v) = m(a+b,v)+\Delta(\delta v) = m(a+b+\overline{\delta}v,v)$ with $\delta := \det(r(a,b,v))$. □

Proposition 2.2. *The following hold.*

(1) *The collection* $\mathcal{M} := \{A[v] \mid [v]\in\mathbf{PG}(V)\}\cup\{A\}$ *is a DHO of rank* 3 *over* $\mathbb{F}_4$ *with ambient space* $S^2(V)$.

(2) *The intersection of* A *with* $A[v]$ *for* $[v]\in\mathbf{PG}(V)$ *is a* 1*-dimensional subspace spanned by* $\Delta(v)$.

(3) *The intersection of* $A[v]$ *with* $A[w]$ *for distinct points* $[v]$ *and* $[w]$ *in* $\mathbf{PG}(V)$ *is a* 1*-dimensional subspace spanned by* $m(w,v) = m(v,w)$.

(4) *Every member of* $\mathcal{M}$ *is totally isotropic with respect to the unitary form* $(\ ,\ )$ *on* $S^2(V)$ *defined above.*

Proof. By the bijective linearity of the map sending $x\in V$ to $\Delta(x)\in A$, the member A is a 3-dimensional subspace of $S^2(V)$. By Lemma 2.1, the subset $A[v]$ for each $[v]\in\mathbf{PG}(V)$ is a subspace of $S^2(V)$. Comparing the off-diagonal components of $m(a,v)$ and $m(b,v)$, the condition $m(a,v) = m(b,v)$ implies that $(a\times v)_i = (b\times v)_i$ for all $i\in\{0,1,2\}$. Then $a\times v = b\times v$, from which we have $(a+b)\times v = 0$, and hence $a+b = \alpha v$ for some $\alpha\in\mathbb{F}_4$. Then $\det(r(a,b,v)) = 0$ and $0 = m(a,v)+m(b,v) = m(a+b,v) = \alpha\Delta(v)$ by equation (11), which implies that $\alpha = 0$ and $a = b$. Thus the map sending $a\in V$ to $m(a,v)\in A[v]$ is injective. Then $A[v]$ is a 3-dimensional subspace of $S^2(V)$ for each $[v]\in\mathbf{PG}(V)$. As the sum $A + A[e_i]$ ($i\in\{0,1,2\}$) contains vectors $x\otimes e_i$ for all $x\in V$, the ambient space of $\mathcal{M}$ contains all basis vectors for $S^2(V)$, and thus it coincides with $S^2(V)$.

We next show that two distinct members of $\mathcal{M}$ intersect at a 1-dimensional subspace. It is immediate to see that $A\cap A[v]$ is a 1-dimensional subspace spanned by $\Delta(v)$ for every $[v]\in\mathbf{PG}(V)$. Take two distinct projective points $[v],[w]$ in $\mathbf{PG}(V)$. Then $m(w,v) = m(v,w)$ lies in both $A[v]$ and $A[w]$ in view of equation (10). Assume that $c := m(x,v) = m(y,w)$ ($x,y\in V$) is a nonzero vector in $A[v]\cap A[w]$. Then, comparing the off-diagonal components of $m(x,v)$ and $m(y,w)$, we have $(x\times v)_i = (y\times w)_i$ for all $i\in\{0,1,2\}$, and thus $x\times v = y\times$

$w =: a$. If $a = 0$, then $x \in [v]$ and $y \in [w]$. This implies that $c = \alpha\Delta(v) = \beta\Delta(w)$ for some $\alpha, \beta \in \mathbb{F}_4$ so that $\alpha = \beta = 0$ as $[v] \neq [w]$, which contradicts that $c \neq 0$. Thus a is a nonzero vector of V. Then the subspace $a^\perp := \{z \in V \mid z \cdot a = 0\}$ is a 2-dimensional subspace of V, which is spanned by v and w, as $[v] \neq [w]$ (see the remark after equation (5)). Thus the vectors x and y lying in $a^\perp$ (by equation (4)) are written as $x = \alpha v + \beta w$ and $y = \gamma v + \delta w$ for some elements α, β, γ, δ in $\mathbb{F}_4$. Then we have

$$\begin{aligned} m(x, v) &= m(\alpha v + \beta w, v) \\ &= m(\alpha v, v) + m(\beta w, v) + \Delta(\det(r(\alpha v, \beta w, v))v) \\ &= \alpha\Delta(v) + \beta m(w, v) \end{aligned}$$

by equation (11). Similarly, we have $m(y, w) = \gamma m(v, w) + \delta\Delta(w)$. Thus $c = m(x, v) = m(y, w)$ implies that

$$(\beta + \gamma)m(v, w) = \Delta(\overline{\alpha}v + \overline{\delta}w).$$

As $[v] \neq [w]$, we have $v \times w \neq 0$, and therefore there is a nonzero off-diagonal component in $m(v, w)$. Thus the last equation holds if and only if $\beta + \gamma = 0$ and $\overline{\alpha}v + \overline{\delta}w = 0$, or equivalently $\beta = \gamma$ and $\alpha = \delta = 0$. Thus $c = m(x, v) = m(y, w) = \beta m(v, w)$. This verifies that $A[v] \cap A[w]$ is a 1-dimensional subspace spanned by $m(w, v)$. We also verified claims (2) and (3).

For distinct projective points $[v]$ and $[w]$ in $\mathbf{PG}(V)$, we have $A \cap A[v] \cap A[w] = \{0\}$, as $A \cap A[v]$ and $A \cap A[w]$ are spanned by $\Delta(v)$ and $\Delta(w)$ respectively. For some pairwise distinct projective points $[v]$, $[w]$ and $[u]$ in $\mathbf{PG}(V)$, suppose $A[v] \cap A[w] \cap A[u]$ is not the zero space. By the conclusion in the above paragraph, $A[v] \cap A[w]$ and $A[v] \cap A[u]$ are spanned respectively by $m(w, v)$ and $m(u, v)$. Thus the assumption implies that $m(w, v) = \alpha m(u, v) = m(\alpha u, v)$ for some nonzero $\alpha \in \mathbb{F}_4$. Recalling that the map sending $a \in V$ to $m(a, v) \in A[v]$ is bijective, this implies that $w = \alpha u$, which contradicts that $[w] \neq [u]$. Hence any three mutually distinct members of $\mathcal{M}$ intersect at the zero space. As there are exactly $22 = ((4^3 - 1)/(4 - 1)) + 1$ members in $\mathcal{M}$, claim (1) is now established.

It remains to show claim (4). By the definition of the unitary form $(\ ,\ )$, A is totally isotropic. Take any projective point $[v]$ in $\mathbf{PG}(V)$ and any vector $m(x, v)$ in $A[v]$. By equation (9), $(m(x, v), m(x, v))$ equals

$$\sum_{i=0}^{2} \{(x \times v)_i(\overline{x_i v_i} + (x \times v)_j(x \times v)_k) + \overline{(x \times v)_i}(x_i v_i + \overline{(x \times v)_j(x \times v)_k})\},$$

which is the sum of $K := \sum_{i=0}^{2}(x \times v)_i\overline{x_i v_i} + \overline{(x \times v)_0(x \times v)_1(x \times v)_2}$ and $\overline{K}$. As $K = 0$ from equation (7), we conclude that $(m(x, v), m(x, v)) = 0$. Thus $M[v]$ is totally isotropic. $\square$

2.3 Automorphisms

The construction in the previous section 2.2 makes easy to find explicit shapes of automorphisms, especially those stabilizing the member A. In this section, we shall determine the linear part of the automorphism group of $\mathcal{M}$. Recall that an *automorphism* (resp. *linear automorphism*) of $\mathcal{M}$ is a semi-linear (resp. linear) bijection on $S^2(V)$ which permutes the members of $\mathcal{M}$. We denote by $\mathrm{Aut}(\mathcal{M})$ the group of all automorphisms of $\mathcal{M}$, in which $L(\mathcal{M})$ denotes the subgroup of all linear automorphisms of $\mathcal{M}$. We shall denote by $x\lambda$ the image of a vector x in a vector space W (V or $S^2(V)$) by a linear map L on W, instead of the usual notation x^L or $L(x)$. This convention will be adopted throughout the remaining part of this paper.

The following result was obtained in [6] as well, but a proof based on the presentation of $\mathcal{M}$ in Section 2.2 is given here:

Lemma 2.3. *Every linear automorphism λ of $\mathcal{M}$ preserves the unitary form $(\ ,\ )$ on $S^2(V)$: $(u\lambda, v\lambda) = (u, v)$ for all $u, v \in S^2(V)$.*

Proof. A member $A\lambda$ of $\mathcal{M}$ is totally isotropic by Proposition 2.2(iv). In particular, $(\Delta_i\lambda, \Delta_j\lambda) = 0$ for all $i, j \in \{0, 1, 2\}$. As $\Delta_i = m(e_i, e_i)$, $\nabla_j = e_i \otimes e_k = m(e_i, e_k)$ and $\nabla_k = m(e_i, e_j)$ are contained in a member $A[e_i]$ for $i \in \{0, 1, 2\} = \{i, j, k\}$, we have $(\Delta_i\lambda, \nabla_j\lambda) = 0$ and $(\nabla_j\lambda, \nabla_k\lambda) = 0$ inside a totally isotropic space $A[e_i]\lambda$ for any pairwise distinct indices i, j, k in $\{0, 1, 2\}$. Thus $(\Delta_i\lambda, \nabla_i\lambda) =: \gamma_i$ ($\in \mathbb{F}_4$) is not zero by the non-degeneracy of $(\ ,\ )$.

Take any i, j, k with $\{i, j, k\} = \{0, 1, 2\}$. As $\iota(e_i \times (e_j + e_k)) = \iota(e_k + e_j) = e_i$ and $\iota(e_k \times (e_j + e_k)) = \iota(e_i) = 0$, we have $m(e_i, e_j + e_k) = e_i \otimes (e_j + e_k) + \Delta(e_i) = \Delta_i + \nabla_j + \nabla_k$ and $m(e_k, e_j + e_k) = e_k \otimes (e_j + e_k) + \Delta(0) = \nabla_i + \Delta_k$. Inside a totally isotropic subspace $A[e_j + e_k]\lambda$, we have $0 = (m(e_i, e_j + e_k)\lambda, m(e_i, e_j + e_k)\lambda)$, which is also calculated as $(\Delta_i\lambda + \nabla_j\lambda + \nabla_k\lambda, \nabla_i\lambda + \Delta_k\lambda) = \gamma_i + \overline{\gamma_k}$. As this holds for all i, j, k with $\{i, j, k\} = \{0, 1, 2\}$, we have $\gamma_0 = \gamma_1 = \gamma_2 \in \mathbb{F}_2$, and therefore $\gamma_0 = \gamma_1 = \gamma_2 = 1$. Hence λ preserves the unitary form $(\ ,\ )$. $\square$

Before determining the stabilizer of A in $L(\mathcal{M})$, we shall show a result which can be obtained by iterating use of equation (3).

Lemma 2.4. *Let u_i ($i \in \{0, 1, 2\}$) be a basis for V. For vectors $x = \sum_{i=0}^{2} x_i u_i$ and $y = \sum_{i=0}^{2} y_i u_i$ in V with $x_i, y_i \in \mathbb{F}_4$ ($i \in \{0, 1, 2\}$), we have the following formula, where δ denotes the determinant of the matrix $r(u_0, u_1, u_2)$ (see equation (4)) and the indices i, j, k follow the convention $\{0, 1, 2\} = \{i, j, k\}$:*

$$m(x, y) = \sum_{i=0}^{2} (x \times y)_i m(u_j, u_k) + \sum_{i=0}^{2} \{x_i y_i + \overline{\delta}\ \overline{(x \times y)_j (x \times y)_k}\} \Delta(u_i). \tag{13}$$

Proof. As $x \times y = \sum_{i=0}^{2}(x_j y_k + x_k y_j)(u_j \times u_k) = \sum_{i=0}^{2}(x \times y)_i(u_j \times u_k)$, it follows from repeatedly using equation (3) that $\iota(x \times y)$ is the sum of the terms $\sum_{i=0}^{2} \iota((x\times y)_i(u_j \times u_k))$ and $\sum_{i=0}^{2}\{(x\times y)_j(u_i\times u_k)\times(x\times y)_k(u_i\times u_j)\}$. Notice that $\iota(\alpha x) = \overline{\alpha}\iota(x)$ ($\alpha \in \mathbb{F}_4$, $x \in V$) and

$$(u_i \times u_k) \times (u_i \times u_j) = \{(u_i \times u_k)\cdot u_j\}u_i + \{(u_i \times u_k)\cdot u_i\}u_j = \delta u_i$$

by equations (4) and (5). Rewriting the above terms with these remarks, we have $\iota(x \times y) = \sum_{i=0}^{2}\overline{(x \times y)_i}\iota(u_j \times u_k) + \delta\{\sum_{i=0}^{2}(x \times y)_j(x \times y)_k u_i\}$. Then $m(x, y) = x \otimes y + \Delta(\iota(x \times y))$ is equal to the sum of $\sum_{i=0}^{2}(x \times y)_i(u_j \otimes u_k) + \sum_{i=0}^{2} x_i y_i \Delta(u_i)$ and $\sum_{i=0}^{2}(x\times y)_i\Delta(\iota(u_j\times u_k)) + \overline{\delta}\sum_{i=0}^{2}\overline{(x \times y)_j(x \times y)_k}\Delta(u_i)$, which coincides with

$$\sum_{i=0}^{2}(x \times y)_i m(u_j, u_k) + \sum_{i=0}^{2}\{x_i y_i + \overline{\delta(x \times y)_j(x \times y)_k}\}\Delta(u_i). \qquad \square$$

Proposition 2.5. *The stabilizer of the member A in the linear automorphism group $L(\mathcal{M})$ coincides with the group $\{\tilde{g} \mid g \in \mathrm{SL}(V)\}$ isomorphic to $\mathrm{SL}(V)$, where $\tilde{g}$ is a bijective linear transformation on $S^2(V)$ defined as follows for each $g \in \mathrm{SL}(V)$ with $g_a := e_a g = \sum_{i=0}^{2} g_{ai}e_i$ ($g_{ai} \in \mathbb{F}_4$) for $a \in \{0, 1, 2\}$. (We also follow the convention on indices $\{i, j, k\} = \{a, b, c\} = \{0, 1, 2\}$):*

$$\Delta_a\tilde{g} = \sum_{i=0}^{2}\overline{g_{ai}}\Delta_i \quad (a \in \{0, 1, 2\}), \tag{14}$$

$$\begin{aligned}\nabla_c\tilde{g} &= \sum_{i=0}^{2}\{g_{ai}g_{bi} + \overline{(g_a \times g_b)_j(g_a \times g_b)_k}\}\Delta_i + \sum_{i=0}^{2}(g_a \times g_b)_i\nabla_i \\ &= g_a \otimes g_b + \Delta(\iota(g_a \times g_b)).\end{aligned} \tag{15}$$

Furthermore, $m(x, y)\tilde{g} = m(xg, yg)$ and thus $A[x]\tilde{g} = A[xg]$ for any $x, y \in V$.

Proof. Take any linear automorphism λ of $\mathcal{M}$ stabilizing A. Then there is a bijection g on V such that $\Delta(x)\lambda = \Delta(xg)$ for every $x \in V$. As λ is linear on $S^2(V)$, we have $\Delta((x + x')g) = \Delta(x + x')\lambda = \Delta(x)\lambda + \Delta(x')\lambda = \Delta(xg + x'g)$ and $\Delta((\alpha x)g) = \Delta(\alpha x)\lambda = (\overline{\alpha}\Delta(x))\lambda = \overline{\alpha}(\Delta(x)\lambda) = \overline{\alpha}\Delta(xg) = \Delta(\alpha(xg))$ for all $x, x' \in V$ and $\alpha \in \mathbb{F}_4$, whence g is linear on V.

The subsequent arguments are based on the following idea. Observe that for each nonzero vector x in V, λ sends the unique member $A[x]$ in $\mathcal{M} \setminus \{A\}$ containing $\Delta(x)$ to the unique member $A[xg]$ in $\mathcal{M} \setminus \{A\}$ containing $\Delta(x)\lambda = \Delta(xg)$. Then $(A[x]\cap A[y])\lambda = A[x]\lambda\cap A[y]\lambda = A[xg]\cap A[yg]$ for distinct projective points $[x]$ and $[y]$ of $\mathbf{PG}(V)$, and therefore a vector $m(x, y)$ spanning $A[x]\cap A[y]$ is sent by λ to a scalar multiple of $m(xg, yg)$ spanning $A[xg] \cap A[yg]$.

We set $g_a := e_a g = \sum_{i=0}^{2} g_{ai} e_i$ for every $a \in \{0,1,2\}$. Then g_a $(a \in \{0,1,2\})$ form a basis for V, and $\Delta_a \lambda = \Delta(e_a g) = \Delta(g_a)$. Take distinct $a, b \in \{0,1,2\}$. Then $m(e_a, e_b) = e_a \otimes e_b$. By the basic idea we described above, $A[e_a]\lambda = A[g_a]$ and we have

$$(e_a \otimes e_b)\lambda = \gamma_{ab} m(g_a, g_b) = \gamma_{a,b}\left(g_a \otimes g_b + \Delta(\iota(g_a \times g_b))\right) \tag{16}$$

for some nonzero element γ_{ab} of $\mathbb{F}_4$. Now we use the fact that λ preserves the unitary form $(\ ,\ )$ (see Lemma 2.3). Take an index $c \in \{0,1,2\}$. Then

$$(\ (e_a \otimes e_b)\lambda, \Delta_c \lambda\ ) = (\ e_a \otimes e_b, \Delta_c\ ) = (\nabla_c, \Delta_c) = 1.$$

As $\Delta_c \lambda$ $(\in A)$ is perpendicular to any element in A, the left hand side of the last equation is calculated as follows in view of equation (16):

$$\begin{aligned}
(\ (e_a \otimes e_b)\lambda, \Delta_c \lambda\ ) &= (\ \gamma_{ab}(g_a \otimes g_b), \Delta(g_c)\ ) \\
&= \gamma_{ab}\left(\sum_{i=0}^{2}(g_a \times g_b)_i \nabla_i, \sum_{j=0}^{2} \overline{g_{cj}} \Delta_j\right) = \gamma_{ab}\sum_{i=0}^{2}(g_a \times g_b)_i g_{ci} \\
&= \gamma_{ab}(g_a \times g_b) \cdot g_c = \gamma_{ab} \det(r(g_a, g_b, g_c)),
\end{aligned}$$

where we use equation (4) in the last line. Here notice that

$$\det(r(g_a, g_b, g_c)) = \det(r(e_a g, e_b g, e_c g)) = \det(r(e_a, e_b, e_c)) \det(g)$$

in view of the definition of $r(x, y, z)$ in (4). Then we have

$$\gamma_{ab} \det(r(e_a, e_b, e_c)) \det(g) = (e_a \otimes e_b, \Delta_c)\,.$$

Taking c such that $\{0,1,2\} = \{a, b, c\}$, we have $\gamma_{ab} = \overline{\det(g)}$.

Thus the action of λ on the basis vectors Δ_i and ∇_i $(i \in \{0,1,2\})$ for $S^2(V)$ is determined as follows: for any $i \in \{0,1,2\} = \{i, j, k\}$,

$$\Delta_i \lambda = \Delta(g_i), \quad \nabla_i \lambda = (e_j \otimes e_k)\lambda = \overline{\det(g)} m(g_j, g_k).$$

Take any distinct projective points $[x]$ and $[y]$ in $\mathbf{PG}(V)$. We denote $x = \sum_{i=0}^{2} x_i e_i$ and $y = \sum_{i=0}^{2} y_i e_i$ with $x_i, y_i \in \mathbb{F}_4$ $(i \in \{0,1,2\})$. Then $m(x, y) = \sum_{i=0}^{2}(x \times y)_i \nabla_i + \sum_{i=0}^{2}\{x_i y_i + \overline{(x \times y)_j (x \times y)_k}\}\Delta_i$ by definition of $m(x, y)$ (see equation (8)), and therefore $m(x, y)\lambda$ is expressed as

$$\sum_{i=0}^{2} \overline{\det(g)}(x \times y)_i m(g_j, g_k) + \sum_{i=0}^{2}\{x_i y_i + \overline{(x \times y)_j (x \times y)_k}\}\Delta(g_i) \tag{17}$$

by the action of λ on the basis Δ_i, ∇_i ($i \in \{0,1,2\}$). On the other hand, as g_i ($i \in \{0,1,2\}$) form a basis for V, Lemma 2.4 implies that $m(xg, yg)$ equals

$$\begin{aligned} m\Big(\sum_{i=0}^{2} x_i g_i, \sum_{i=0}^{2} y_i g_i\Big) \\ = \sum_{i=0}^{2}(x \times y)_i m(g_j, g_k) + \sum_{i=0}^{2}\{x_i y_i + \overline{\delta}\ \overline{(x \times y)_j (x \times y)_k}\}\Delta(g_i) \end{aligned} \tag{18}$$

with $\delta = \det(r(g_0, g_1, g_2)) = \det(g)$. Now recall that $m(x,y)\lambda = \gamma_{x,y} m(xg, yg)$ for some nonzero element $\gamma_{x,y}$ in $\mathbb{F}_4$, because $(A[x] \cap A[y])\lambda = A[xg] \cap A[yg]$ and $m(x,y)$ and $m(xg, yg)$ respectively span $A[x] \cap A[y]$ and $A[xg] \cap A[yg]$.

As $\Delta(g_i)$ and $m(g_j, g_k)$ ($i \in \{0,1,2\} = \{i,j,k\}$) form a basis for $S^2(V)$, this equation with equation (17) and equation (18) implies that

$$\begin{aligned} \overline{\det(g)}(x \times y)_i = \gamma_{xy}(x \times y)_i, \quad \text{and} \\ x_i y_i + \overline{(x \times y)_j (x \times y)_k} = \gamma_{xy}\{x_i y_i + \overline{\det(g)(x \times y)_j (x \times y)_k}\} \end{aligned}$$

for all $i \in \{0,1,2\}$. As $[x] \neq [y]$, there exists $i \in \{0,1,2\}$ with $(x \times y)_i \neq 0$. Thus we have $\gamma_{xy} = \overline{\det(g)}$ from the first equation above. Then the second equation above reads that for all $i \in \{0,1,2\}$ we have

$$(x_i y_i)(1 + \overline{\det(g)}) = (1 + \det(g))\overline{(x \times y)_j (x \times y)_k}.$$

Recall that this conclusion holds for every distinct projective points $[x]$ and $[y]$ of $\mathbf{PG}(V)$. Take $x = e_0 + e_1 + e_2$ and $y = e_0 + \omega e_1 + \overline{\omega} e_2$. Then we can verify that $x \times y = y$, and thus $x_0 y_0 = 1$ and $(x \times y)_1 (x \times y)_2 = 1$. Hence the above conclusion for these x, y reads $1 + \overline{\det(g)} = 1 + \det(g)$, whence $\det(g) = \overline{\det(g)} = 1$. Thus we showed that if λ is a linear automorphism of $\mathcal{M}$, then λ is of the form $\tilde{g}$ for some $g \in \mathrm{SL}(V)$.

Conversely, for every $g \in \mathrm{SL}(V)$, we define $\lambda := \tilde{g}$ to be a linear map defined by equations (14) and (15). Then it follows from equations (17) and (18) that $m(x,y)\lambda = m(xg, yg)$ for all $x, y \in V$ with $[x] \neq [y]$. As $m(x,x)\lambda = \Delta(x)\lambda = \sum_{i=0}^{2} \overline{x_i}\Delta_i\lambda = \sum_{i=0}^{2} \overline{x_i}\Delta(g_i) = \Delta(xg)$, the equation $m(x,y) = m(xg, yg)$ holds for all $x, y \in V$, and therefore $\lambda = \tilde{g}$ is in fact a linear automorphism of $\mathcal{M}$. $\square$

The same idea described in the second paragraph of the proof for Proposition 2.5 can also be used to find an automorphism which moves A.

Lemma 2.6. *Let σ be a linear bijection on $S^2(V)$ which fixes ∇_2 and Δ_2 and interchanges the pairs (∇_i, Δ_i) for $i = 0$ and 1. Then σ is an automorphism* *of $\mathcal{M}$. Moreover $\Delta(x)\sigma = m(e_2, \delta(x))$ and $m(x,y)\sigma = m(\delta(x), \delta(y))$, and hence $A\sigma = A[e_2]$, $A[e_2]\sigma = A$, $A[x]\sigma = A[\delta(x)]$ for every $x, y \in V \setminus [e_2]$, where*

$$\delta(x) := \overline{x_1}e_0 + \overline{x_0}e_1 + (x_0x_1 + \overline{x_2})e_2. \tag{19}$$

Proof. Assume that σ is an automorphism of $\mathcal{M}$. Then σ interchanges A and $A[e_2]$, as A and $A[e_2]$ are the unique members of $\mathcal{M}$ containing $\Delta(e_2) = \Delta(e_2)\sigma$. Thus for each $x = \sum_{i=0}^{2} x_ie_i$ in V, we have $\Delta(x)\sigma = m(e_2, y)$ for some $y = \sum_{i=0} y_ie_i \in V$. We have $m(e_2, y) = e_2 \otimes y + \overline{y_0y_1}\Delta_2 = y_0\nabla_1 + y_1\nabla_0 + (y_2 + \overline{y_0y_1})\Delta_2$, as $\iota(e_2, y) = y_0y_1e_2$. On the other hand, we have $\Delta(x)\sigma = \sum_{i=0} \overline{x_i}\Delta_i\sigma = \overline{x_0}\nabla_0 + \overline{x_1}\nabla_1 + \overline{x_2}\Delta_2$. Thus, comparing the coefficients, we conclude that $y = \overline{x_1}e_0 + \overline{x_0}e_1 + (x_0x_1 + \overline{x_2})e_2 = \delta(x)$ (see equation (19)) satisfies $\Delta(x)\sigma = m(e_2, \delta(x))$. In particular, $A[x]\sigma = A[\delta(x)]$ for each point $[x]$ in $\mathbf{PG}(V)$ distinct from $[e_2]$, because $A[e_2]$, $A[\delta(x)]$ and $A[x]\sigma$ are members of $\mathcal{M}$ containing $(A \cap A[x])\sigma = A[e_2] \cap A[x]\sigma = A[e_2] \cap A[\delta(x)]$.

For two distinct points $[x]$ and $[y]$ in $\mathbf{PG}(V) \setminus \{[e_2]\}$, the nonzero vector $m(x,y)$ in $A[x] \cap A[y]$ is sent by σ to $A[x]\sigma \cap A[y]\sigma = A[\delta(x)] \cap A[\delta(y)]$, which is spanned by $m(\delta(x), \delta(y))$. Hence there is a nonzero element $\alpha \in \mathbb{F}_4$ such that $\alpha(m(x,y)\sigma) = m(\delta(x), \delta(y))$. Comparing the coefficients of $\nabla(e_i)$ and $\Delta(e_i)$ for $i \in \{0,1,2\}$ with some calculations, we obtain the following equations:

$$\begin{aligned}
\alpha\{x_0y_0 + \overline{(x\times y)_1(x\times y)_2}\} &= x_0y_0(x\times y)_2 + \overline{(x\times y)_1},\\
\alpha\{x_1y_1 + \overline{(x\times y)_0(x\times y)_2}\} &= x_1y_1(x\times y)_2 + \overline{(x\times y)_0},\\
\alpha(x\times y)_2 &= \overline{(x\times y)_2},\\
\alpha(x\times y)_0 &= \overline{x_1y_1}\{1 + (x\times y)_2^3\} + (x\times y)_0(x\times y)_2,\\
\alpha(x\times y)_1 &= \overline{x_0y_0}\{1 + (x\times y)_2^3\} + (x\times y)_1(x\times y)_2,\\
\alpha\{x_2y_2 + \overline{(x\times y)_0(x\times y)_1}\} &= \overline{(x_2y_2 + x_0y_1)}(1 + (x\times y)_2^3)\\
&\quad + \{x_2y_2 + \overline{(x\times y)_0(x\times y)_1)}\}(x\times y)_2.
\end{aligned}$$

Conversely, in order to verify that σ lies in $L(\mathcal{M})$, it suffices to show that for all distinct points $[x]$ and $[y]$ in $\mathbf{PG}(V) \setminus \{[e_2]\}$ there exists some nonzero element α in $\mathbb{F}_4$ such that $m(\alpha x, y)\sigma = m(\delta(x), \delta(y))$. (Notice that this implies $A[x]\sigma = A[\delta(x)]$.) It is immediate to see that this holds for $\alpha = (x \times y)_2$, if $(x\times y)_2 \neq 0$. When $(x \times y)_2 = 0$ and so $(x\times y)_i \neq 0$ for some $i \in \{0,1\} = \{i, j\}$, we can verify that $\overline{x_jy_j(x\times y)_i}$ can be taken as α, by somewhat complicated calculations. (In fact, we have several expressions for α, which turn out to be identical, as all x_i, y_i are contained in $\mathbb{F}_4$.) $\square$

From Proposition 2.5 with Lemma 2.6 and Lemma 2.3, $L(\mathcal{M})$ is a subgroup of the unitary group $U(S^2(V))$ ($\cong \mathrm{GU}_6(2)$) of order $|\mathcal{M}|\cdot|\mathrm{SL}(V)| = 22\cdot 3\cdot 2^6.3^2.5.7$, which induces a triply transitive action on the members of $\mathcal{M}$. The kernel of the action is a subgroup Z of order 3 generated by $\tilde{\omega I}$ with the notation in Proposition 2.5. As the factor group of the stabilizer of A by Z is isomorphic to a simple group $\mathrm{PSL}_3(4)$, it is standard in group theory to show that $L(\mathcal{M})/Z$ is a simple group of order $2^7.3^2.5.7.11$. The field automorphism ϕ of $S^2(V)$ sending $\sum_{i=0}^{2}(x_i\Delta_i + y_i\nabla_i)$ to $\sum_{i=0}^{2}(\overline{x_i}\Delta_i + \overline{y_i}\nabla_i)$ is a semilinear automorphism of $\mathcal{M}$. As the semilinear automorphism group of $S^2(V)$ is generated by the linear automorphisms and ϕ, $\mathrm{Aut}(\mathcal{M}) = L(\mathcal{M})\,\langle\phi\rangle$. Summarizing, we have:

Proposition 2.7. *Let $\mathcal{M}$ be the DHO constructed in Section 2.2. Then the linear automorphism group $L(\mathcal{M})$ is a subgroup of the special unitary group $\mathrm{SU}_6(2)$. The factor group of $L(\mathcal{M})$ by its center Z of order 3 is a simple group of order $2^7.3^2.5.7.11$ acting as a triply transitive permutation group on the 22 members of $\mathcal{M}$. The automorphism group $\mathrm{Aut}(\mathcal{M})$ contains $L(\mathcal{M})$ of index 2. There are exactly $|\Gamma\mathrm{L}_6(4)|/|\mathrm{Aut}(\mathcal{M})|$ ($= 2^{23}.3^5.5^2.7.13.17.31$) DHOs isomorphic to $\mathcal{M}$.*

2.3.1 Steiner system $S(22, 6, 3)$ on $\mathcal{M}$

The simple group $L(\mathcal{M})/Z$ is in fact isomorphic to the Mathieu group M_{22}. This fact can be shown, for example, by a characterization of a simple group of order $2^7.3^2.5.7.11$. If we adopt a usual definition of M_{22} as the automorphism group of a Steiner system $S(22, 6, 3)$, this fact $L(\mathcal{M})/Z \cong M_{22}$ can also be established by constructing a Steiner system $S(22, 6, 3)$ on which $L(\mathcal{M})/Z$ acts faithfully, because $S(22, 6, 3)$'s form a single isomorphism class.

We shall briefly describe a construction of such a system. The key observation is as follows (which was first observed by Del Fra in [2] in a more general setting): for any 3-subset T of $\mathcal{M}$, there is a unique 6-subset $B(T)$ of $\mathcal{M}$ containing T such that $X \cap X'$ with X' ranging over $B(T) \setminus \{X\}$ lie on a common 2-subspace for any $X \in T$. This can be verified for $T = \{A, A[e_1], A[e_2]\}$, and therefore for any 3-subset T of $\mathcal{M}$ by the triple transitivity of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. The above property of $B(T)$ implies that $B(T) = B(T')$ for any 3-subset T' contained in $B(T)$. Defining $\mathcal{B}$ as a collection of 6-subsets $B(T)$ for all 3-subsets T of $\mathcal{M}$, it is immediate to see that a system $(\mathcal{M}, \mathcal{B})$ is a Steiner system $S(22, 6, 3)$ on which $L(\mathcal{M})/Z$ acts faithfully.

2.3.2 Remark

We give a remark on the matrix shape of a linear automorphism $\tilde{g}$ of $\mathcal{M}$ obtained from a linear map g in $\mathrm{SL}(V)$ using Proposition 2.5. We identify g with the matrix representing g with respect to the basis e_0, e_1, e_2 for V. We introduce some notation: For any matrix h of degree 3 whose (a, i)-entry is h_{ai} $(a, i \in$

$\{0,1,2\}$), $\overline{h}$ denotes the matrix whose (a,i)-entry is $\overline{h_{ai}} = h_{ai}^2$. Furthermore, $\iota(h)$ denotes the matrix whose (a,i)-entry is $h_{aj}h_{ak}$ for indices j,k with $\{i,j,k\} = \{0,1,2\}$. Thus $\iota(h)$ is the matrix obtained from h by applying the operation ι introduced in Section 2.1 to all rows of h. With this notation, the matrix representing $\tilde{g}$ with respect to Δ_i, ∇_i $(i=0,1,2)$ is:

$$\begin{pmatrix} \overline{g} & 0 \\ L(g) & {}^t g^{-1} \end{pmatrix}, \quad \text{with } L(g) := {}^t(\iota({}^t g)) + \iota({}^t(\overline{g})^{-1}).$$

This is immediately obtained from equations (14) and (15), noticing the following facts: $(g_b \times g_c)_i = g_{bj}g_{ck} + g_{bk}g_{cj}$ $(\{a,b,c\} = \{i,j,k\} = \{0,1,2\})$ is the determinant of the matrix obtained from g by deleting the a-th row and the i-th column, whence it is the (i,a)-entry of the so called the adjugate matrix of g; the adjugate matrix of a matrix g with $\det(g) = 1$ coincides with g^{-1}. The above formula is convenient to calculate $\tilde{g}$ for a matrix $g \in \mathrm{SL}(V)$ specifically when $g_{ij} \neq 0$ for all $i,j \in \{0,1,2\}$. This also shows that the $L(\mathcal{M})_A$ $(\cong \mathrm{SL}(V))$-module $S^2(V)/A$ is the Frobenius twist of the dual module to A. As the map sending $g \in \mathrm{SL}(V)$ to $\tilde{g} \in L(M)_A$ is an isomorphism, we should have $L(gh) = L(g)\overline{h} + ({}^t g^{-1})L(h)$ for all $g,h \in \mathrm{SL}(V)$. This can be obtained from the following formula for any matrices a,b of degree 3, which can be verified by direct calculations:

$$\iota(ab) = \overline{a}\iota(b) + \iota(a)({}^t b^{-1}).$$

2.4 Comments about the construction and automorphisms

Del Fra [2] showed the uniqueness up to isomorphism of a 2-(projective) dimensional DHO over $\mathbb{F}_4$ with a certain condition, which is satisfied by the DHO $\mathcal{M}$. Nakagawa showed the uniqueness of a unitary DHO of rank 3 over $\mathbb{F}_4$ with ambient space U of dimension 6 [6]. He also gives a precise description of all members of the Mathieu DHO, and showed that its full automorphism group contains the triple central cover of M_{22} as a subgroup of index 2.

Thanks to the above-mentioned result by Nakagawa, it follows from Proposition 2.2 that our DHO $\mathcal{M}$ is isomorphic to the Mathieu DHO. In fact, without much effort, we can verify that the above member A and $A[v]$ of $\mathcal{M}$ are respectively identical to his member A and $A[v]$, if we identify Δ_i and ∇_i $(i=0,1,2)$ with $\mathbf{e}_i$ and $\mathbf{e}_{5-i}$ $(i=0,1,2)$. Thus, in this sense, the construction above just gives another presentation of a well-known model of the Mathieu DHO. However, observe that our construction is simple and uniform, and that it makes easy to find the intersections of two given members (this is a bit messy for members of type $A[\alpha\theta,\alpha,1]$ with notation in [6]). Furthermore, equation (8) shows that $\mathcal{M}$ is a deformation of the Veronesean DHO $\mathcal{V}_3(\mathbb{F}_4)$, because $x \times y$ $(y \neq 0)$ is a typical vector in a member $V[y]$ of $\mathcal{V}_3(\mathbb{F}_4)$.

Proposition 2.7 has been essentially shown in [6] as well. One advantage of our approach via the DHO $\mathcal{M}$ is that we obtain rather explicit unitary matrices of degree 6 generating $L(\mathcal{M})$. These matrices will be effectively used to classify certain subspaces of $S^2(V)$ (see Section 3.4).

3 Complements to the Mathieu DHO

3.1 Splitness of the Mathieu DHO

We use the notation and convention in Section 2.1. Let $\mathcal{M}$ be the model of the Mathieu DHO constructed in Section 2.2. Recall that a *complement* to $\mathcal{M}$ is a subspace Y of $S^2(V)$ with $S^2(V) = X \oplus Y$ for every member X of $\mathcal{M}$. We shall interpret the existence of a complement in terms of some alternating semi-bilinear maps, and then show that $\mathcal{M}$ is of split type.

Lemma 3.1. *Any complement to $\mathcal{M}$ coincides with*

$$Y(g) := \{\Delta(g(x,y)) + \sum_{i=0}^{2}(x \times y)_i \nabla_i \mid x, y \in V\}$$

for a map g from $V \times V$ to V which satisfies the following conditions.

(i) $g(x,x) = 0$, $g(\alpha x, y) = \overline{\alpha} g(x,y)$, $g(x + x', y) = g(x,y) + g(x', y)$ *for all* $x, x', y \in V$ *and* $\alpha \in \mathbb{F}_4$.

(ii) *for every nonzero vector* $a, b \in V$ *with* $[a] \neq [b]$,

$$\sum_{i=0}^{2} \overline{(a \times b)_i} g(e_j, e_k) \neq \sum_{i=0}^{2} \{\overline{a_i b_i} + (a \times b)_j (a \times b)_k\} e_i,$$

where the indices i, j, k follow the convention that $\{i, j, k\} = \{0, 1, 2\}$.

Proof. We first show that a 3-dimensional subspace Y of $S^2(V)$ intersects $A = \{\Delta(x) \mid x \in V\}$ trivially if and only if $Y = Y(g)$ for a map g from $V \times V$ to V satisfying condition (i) in Lemma. Assume that Y is a subspace of $S^2(V)$ with $A \oplus Y = S^2(V)$. Then the vector $m(x,y)$ $x, y \in V$ in equation (8) is uniquely written as $m(x,y) = \Delta(f(x,y)) + \gamma(x,y)$ with $f(x,y) \in V$, $\gamma(x,y) \in Y$.

We denote by $\Delta(g(x,y))$ and $\nabla(x,y)$ respectively the sums of the diagonal and off-diagonal components of $\gamma(x,y)$. Then we have $\gamma(x,y) = \Delta(g(x,y)) + \nabla(x,y)$. In view of the diagonal and off-diagonal components of $m(x,y) = \Delta(f(x,y) + g(x,y)) + \nabla(x,y)$, we have

$$\nabla(x,y) = \textstyle\sum_{i=0}^{2}(x \times y)_i \nabla_i,$$
$$f(x,y) + g(x,y) = \textstyle\sum_{i=0}^{2}\{\overline{x_i y_i} + (x \times y)_j (x \times y)_k\} e_i.$$

As $m(x,y)=m(y,x)$, $m(x,x)=\Delta(x)$ and $m(\alpha x,y)=\alpha m(x,y)$ for all $x,y\in V$ and $\alpha\in\mathbb{F}_4$, we have

$$f(x,y)=f(y,x),\quad f(x,x)=x,\quad f(\alpha x,y)=\overline{\alpha}f(x,y);$$
$$g(x,y)=g(y,x),\quad g(x,x)=0,\quad g(\alpha x,y)=\overline{\alpha}g(x,y).$$

As $m(x+x',y)+m(x,y)+m(x',y)=\Delta(\delta y)$ with $\delta=\det(r(x,x',y))$ for $x,x',y\in V$ (see equation (11)),

$$\Delta\left(f(x+x',y)+f(x,y)+f(x',y)+\delta y\right)=\gamma(x+x',y)+\gamma(x,y)+\gamma(x',y).$$

The left hand side of this equation lies in A, while the right hand side lies in Y. As $A\cap Y=\{0\}$, this implies that $f(x+x',y)=f(x,y)+f(x',y)+\delta(r(x,x',y))y$ and $\gamma(x+x',y)=\gamma(x,y)+\gamma(x',y)$. As $\nabla(x,y)=\sum_{i=0}^{2}(x\times y)_i\nabla_i$ is bilinear, we have $g(x+x',y)=g(x,y)+g(x',y)$ for any $x,x',y\in V$. Thus g satisfies the property (i) in Lemma. Furthermore, Y coincides with

$$Y(g):=\{\Delta(g(x,y))+\textstyle\sum_{i=0}^{2}(x\times y)_i\nabla_i \mid x,y\in V\},$$

because $Y(g)$ is a subset of Y which forms a subspace of $S^2(V)$ by property (i) and it contains three independent vectors $\Delta(g(e_j,e_k))+\nabla_i$ ($i\in\{0,1,2\}=\{i,j,k\}$).

Conversely, if g is a map from $V\times V$ to V having property (i), then $Y(g)$ defined in Lemma forms a subspace of $S^2(V)$, which is of dimension 3 in view of the off-diagonal parts. We have $Y(g)\cap A=\{0\}$, because if a vector $\Delta(g(x,y))+\sum_{i=0}^{2}(x\times y)_i\nabla_i$ lies in A, then $x\times y=0$, and therefore $x=0$ or $y=\alpha x$ for some $\alpha\in\mathbb{F}_4$, which implies $g(x,y)=0$.

Thus in order to verify Lemma it suffices to show that $Y(g)\cap A[b]=\{0\}$ for any projective point $[b]$ in $\mathbf{PG}(V)$ if and only if g satisfies the property (ii). There is a nonzero vector $b\in V$ such that $Y(g)\cap A[b]\neq\{0\}$ if and only if there are vectors $x,y,a\in V$ such that $a\neq 0$ and $[a]\neq[b]$ and $m(a,b)=\Delta(g(x,y))+\sum_{i=0}^{2}(x\times y)_i\nabla_i$. Comparing the off-diagonal and diagonal components of the both sides of this equation, this is equivalent to the following conditions:

$$a\times b=x\times y\,(\neq 0)\quad\text{and}\quad g(x,y)=\textstyle\sum_{i=0}^{2}\{\overline{a_ib_i}+(a\times b)_j(a\times b)_k\}e_i.$$

Then x and y are nonzero vectors lying in $\langle a,b\rangle=(a\times b)^{\perp}$, the hyperplane of V perpendicular to $a\times b$ with respect to the dot product. Thus there are α, β, γ and δ in $\mathbb{F}_4$ such that $x=\alpha a+\beta b$ and $y=\gamma a+\delta b$. As $0\neq a\times b=x\times y=(\alpha\delta+\beta\gamma)(a\times b)$, we have $\alpha\delta+\beta\gamma=1$. As g satisfies the property (i), we have $g(x,y)=\sum_{i=0}\overline{(x\times y)_i}g(e_j,g_k)=\sum_{i=0}^{2}\overline{(a\times b)_i}g(e_j,e_k)$. Thus there is no nonzero $b\in V$ with $Y(g)\cap A[b]\neq\{0\}$ if and only if the property (ii) holds. $\square$

Proposition 3.2. *The Mathieu DHO $\mathcal{M}$ is of split type: the subspace $Y(g)$ in Lemma 3.1 is a complement to $\mathcal{M}$ for the map $g(x,y) = \overline{\omega} \sum_{i=0}^{2} \overline{(x \times y)_i} e_i$.*

Proof. Take a map g from $V \times V$ to V given by $g(x,y) := \overline{\omega} \sum_{i=0}^{2} \overline{(x \times y)_i} e_i$ for $x, y \in V$. Notice that $g(e_j, e_k) = \overline{\omega} e_i$ for indices i, j, k with $\{0,1,2\} = \{i,j,k\}$. As g satisfies condition (i) in Lemma 3.1, it suffices to verify that g satisfies condition (ii).

Suppose there are some nonzero $a, b \in V$ with $[a] \neq [b]$ (or equivalently $a \times b \neq 0$) such that $\sum_{i=0}^{2} \overline{(a \times b)_i} \overline{\omega} e_i = \sum_{i=0}^{2} \{\overline{a_i b_i} + (a \times b)_j (a \times b)_k\} e_i$. Thus $d_i := \overline{\omega} \overline{(a \times b)_i} + \overline{a_i b_i} + (a \times b)_j (a \times b)_k = 0$ for all $i \in \{0,1,2\}$. Then we have $\sum_{i=0}^{2} d_i (a \times b)_i = 0$. On the other hand, as $\overline{(a \times b)_i} (a \times b)_i = (a \times b)_i^3 = 0$ or 1 for each $i \in \{0,1,2\}$ and $\sum_{i=0}^{2} \overline{a_i b_i} (a \times b)_i = \overline{\pi}$ with $\pi := (a \times b)_0 (a \times b)_1 (a \times b)_2$ by equation (7), the sum $\sum_{i=0}^{2} d_i (a \times b)_i$ is calculated to be $\overline{\omega}(\sum_{i=0}^{2} (a \times b)_i^3) + \pi + \overline{\pi}$. This value is equal to 0 if and only if exactly one of $(a \times b)_i$'s is 0, because one of the $(a \times b)_i$'s is not 0 and $\overline{\omega}$ does not lie in $\mathbb{F}_2$.

In this case, without loss of generality, we may assume that $(a \times b)_0 = 0$ but $(a \times b)_1 \neq 0$ and $(a \times b)_2 \neq 0$. Then the relation $0 = d_0$ reads $0 = \overline{a_0 b_0} + (a_0 b_2 + b_0 a_2)(a_0 b_1 + b_0 a_1) = \overline{a_0 b_0} + \overline{a_0} b_1 b_2 + \overline{b_0} a_1 a_2 + a_0 b_0 (a_1 b_2 + b_1 a_2) = \overline{a_0 b_0} + \overline{a_0} b_1 b_2 + \overline{b_0} a_1 a_2$, since $(a \times b)_0 = 0$. The conditions $d_i = 0$ for $i = 1, 2$ read $\omega (a \times b)_i = a_i b_i$. In particular, $a_i \neq 0$ and $b_i \neq 0$ for $i \in \{1, 2\}$. If $a_0 = 0$, then the above equation implies that $b_0 = 0$, but this contradicts that $(a \times b)_i \neq 0$ for $i \neq 0$. Thus $a_0 \neq 0$ and similarly $b_0 \neq 0$. Then the above equation is equivalent to the equation $a + b = 1$, where we set $a := a_0 a_1 a_2$ and $b := b_0 b_1 b_2$. As a and b are nonzero elements in $\mathbb{F}_4$, this implies that $\{a, b\} = \{\omega, \overline{\omega}\}$, and thus $ab = 1$. On the other hand, the product of the equations $\omega (a \times b)_i = a_i b_i$ for $i = 1, 2$ is calculated as follows: $a_1 a_2 b_1 b_2 = \overline{\omega}(a_0 b_1 + b_0 a_1)(a_0 b_2 + b_0 a_2) = \overline{\omega}\left(\overline{a_0} b_1 b_2 + a_0 b_0 (a_1 b_2 + a_2 b_1) + \overline{b_0} a_1 a_2\right) = \overline{\omega}(\overline{a_0} b_1 b_2 + \overline{b_0} a_1 a_2)$. Multiplying this equation by $a_0 b_0$, we obtain $ab = \overline{\omega}(b + a) = \overline{\omega}$. This contradicts the above conclusion that $ab = 1$. Thus we verified that the condition (ii) is satisfied by our map g. □

3.2 General property of DHO-sets

In this section, we shall discuss some fundamental facts about the DHO-set for a DHO of split type. In general, assume that a DHO $\mathcal{S}$ over $\mathbb{F}_q$ of rank n splits over a subspace Y of its ambient space U. Choose a member X of $\mathcal{S}$. Then every vector $\mathbf{u}$ of U is uniquely decomposed as $\mathbf{u} = x(\mathbf{u}) + y(\mathbf{u})$ with $x(\mathbf{u}) \in X$ and $y(\mathbf{u}) \in Y$. Identifying $\mathbf{u}$ with the pair $(x(\mathbf{u}), y(\mathbf{u}))$, we have $U = X \oplus Y$.

We can parametrize the members of $\mathcal{S}$ by $\{0\} \cup \mathbf{PG}(X)$ in the following way, where $\mathbf{PG}(X)$ denotes the set of projective points of X. We set $X_0 := X$. For a projective point $[\mathbf{a}] = \mathbb{F}_q\mathbf{a}$ of X, let $X[\mathbf{a}]$ be the unique member of $\mathcal{S}$ such that $X \cap X[\mathbf{a}] = [\mathbf{a}]$. We shall fix this parametrization from now on.

Take a projective point $[\mathbf{a}]$ in $\mathbf{P}G(X)$. Since every vector $\mathbf{u}$ of $X[\mathbf{a}]$ is uniquely written as $\mathbf{u} = x(\mathbf{u})+y(\mathbf{u})$ for some $x(\mathbf{u}) \in X$ and $y(\mathbf{u}) \in Y$, a map $\chi_{[\mathbf{a}]}$ sending $\mathbf{u}$ to $x(\mathbf{u})$ is a well-defined $\mathbb{F}_q$-linear map from $X[\mathbf{a}]$ to X. As $X[\mathbf{a}] \cap Y = \{\mathbf{0}\}$, the kernel of $\chi_{[\mathbf{a}]}$ is $\{\mathbf{0}\}$, and therefore $\chi_{[\mathbf{a}]}$ is a bijection. Then a map $L[\mathbf{a}]$ sending each $\mathbf{x} \in X$ to $y(\chi_{[\mathbf{a}]}^{-1}(\mathbf{x}))$ is an $\mathbb{F}_q$-linear map from X to Y. Furthermore, with the above identification, we have

$$X[\mathbf{a}] = \{(\mathbf{x}, \mathbf{x}L[\mathbf{a}]) \mid \mathbf{x} \in X\} = \{\mathbf{x} + \mathbf{x}L[\mathbf{a}] \mid \mathbf{x} \in X\}$$

for every $[\mathbf{a}] \in \mathbf{PG}(X)$. Defining $L[0]$ as the zero map, this expression holds for every member of $\mathcal{S}$. Notice that we denote the image of a vector $\mathbf{x}$ by a linear map L by $\mathbf{x}L$, instead of the usual notation $\mathbf{x}^L$ or $L(\mathbf{x})$.

The collection $\mathcal{L}_X(\mathcal{S}; Y)$ of linear maps $L[\mathbf{a}]$ ($\mathbf{a} = 0$ or $\langle \mathbf{a} \rangle \in \mathbf{PG}(X)$) indexed by $\tilde{X} := \{0\} \cup \mathbf{PG}(X)$ is called *the DHO-set for $\mathcal{S}$ with respect to a member X and a complement Y*. As we saw above, $X = X(0) = \{x \mid x \in X\}$ and $X[\mathbf{a}] = \{x + xL[\mathbf{a}] \mid x \in X\}$. Recall the following conditions for $\mathcal{S}$ to be a DHO:

(1) $X[\mathbf{a}] \cap X[\mathbf{b}]$ is 1-dimensional for every distinct $[\mathbf{a}], [\mathbf{b}] \in \tilde{X}$,

(2) $X[\mathbf{a}] \cap X[\mathbf{b}] \cap X[\mathbf{c}] = \{0\}$ for pairwise distinct $[\mathbf{a}]$, $[\mathbf{b}]$ and $[\mathbf{c}]$ in $\tilde{X}$.

These conditions can be expressed in terms of the kernels of linear maps $L[\mathbf{a}] + L[\mathbf{b}]$ in the following way:

(1') For every distinct $[\mathbf{a}]$ and $[\mathbf{b}]$ in $\tilde{X}$, the subspace $\{\mathbf{x} \in X \mid \mathbf{x}L[\mathbf{a}] = \mathbf{x}L[\mathbf{b}]\}$ is of dimension 1. We denote by $\kappa([\mathbf{a}], [\mathbf{b}])$ a nonzero vector in the 1-dimensional kernel of $L[\mathbf{a}] + L[\mathbf{b}]$.

(2') For each $[\mathbf{a}]$ in $\tilde{X}$, the map from $\tilde{X} \setminus \{[\mathbf{a}]\}$ to itself sending $[\mathbf{b}]$ to $[\kappa([\mathbf{a}], [\mathbf{b}])]$ is a bijection; or equivalently, $\{\mathbf{x} \in X \mid \mathbf{x}L[\mathbf{a}] = \mathbf{x}L[\mathbf{b}] = \mathbf{x}L[\mathbf{c}]\}$ is the zero space for any mutually distinct $[\mathbf{a}]$, $[\mathbf{b}]$, $[\mathbf{c}]$ in $\tilde{X}$.

Hence any collection of linear maps $L[\mathbf{a}]$ from X to Y parametrized by $\tilde{X} = \{0\} \cup \mathbf{PG}(X)$ gives a DHO of rank n over $\mathbb{F}_q$ (with complement Y and a member X) if it satisfies the above two conditions (1') and (2').

The terminology 'DHO-set' is introduced in [3, Definition 3.2(b)] for a DHO $\mathcal{S}$ over $\mathbb{F}_2$ of split type. In this paper, this terminology is adopted for a general DHO of split type, instead of the terminology 'linear system' used by the author (e.g. [11]).

3.3 DHO-sets for the Mathieu DHO

In this section, we shall describe the DHO-set $\mathcal{L}_A(\mathcal{M}; Y(g))$ for the DHO $\mathcal{M}$ with respect to the member A and the complement $Y(g)$ for an alternating map g defined in Proposition 3.2. We set $\Gamma_i := \omega\Delta_i + \nabla_i$ for $i \in \{0,1,2\}$. Then Γ_i $(i = 0,1,2)$ form a basis for the complement $Y(g)$. Notice that $(\Delta_i, \Gamma_{i'}) = \delta_{i,i'}$ and $(\Gamma_i, \Gamma_{i'}) = \delta_{i,i'}$ for $i, i' \in \{0,1,2\}$, where the symbol $\delta_{i,i'}$ denotes the Kronecker's delta.

Take $[v] \in \mathbf{PG}(V)$. As $\nabla_i = \Gamma_i + \omega\Delta_i$ $(i \in \{0,1,2\})$, a typical vector $m(x,v) = \sum_{i=0}^{2} ((x \times v)_i \nabla_i + z_i \Delta_i)$ in $A[v]$ (with $z_i = x_i v_i + \overline{(x \times v)_j (x \times v)_k}$) is written as $m(x,y) = \sum_{i=0}^{2} \{(x \times v)_i \Gamma_i + (\omega(x \times v)_i + z_i)\,\Delta_i\}$. Thus the decomposition of $m(x,v)$ in the direct sum $A \oplus Y(g)$ is given by $m(x,v) = a(x,v) + a(x,v)L[v]$, where

$$\begin{aligned} a(x,y) &= \textstyle\sum_{i=0}^{2} (\omega(x \times v)_i + x_i v_i + \overline{(x \times v)_j (x \times v)_k})\Delta_i \quad \text{and} \\ a(x,y)L[v] &= \textstyle\sum_{i=0}^{2} (x \times v)_i \Gamma_i. \end{aligned}$$

We shall calculate an explicit form of $L[v]$ above as the matrix $M[v]$ representing $L[v]$ with respect to a basis $(\Delta_i)_{i=0}^{2}$ for A and a basis $(\Gamma_i)_{i=0}^{2}$ for $Y(g)$. We set $M[0] = 0$ and $\tilde{V} := \{[0]\} \cup \mathbf{PG}(V)$. By the discussions in the previous section, $M[v] + M[w]$ is of rank 2 for any distinct $[v]$, $[w]$ in $\tilde{V}$. If the kernel of $M[v] + M[w]$ is denoted by $\kappa([v],[w])$, then the map sending $[w]$ to $\kappa([v],[w])$ is a bijection on $\tilde{V} \setminus \{[v]\}$. This collection of matrices $M[v]$ for $[v] \in \tilde{V}$ also satisfy the following property:

Lemma 3.3. *For each $[v] \in \tilde{V}$, the matrix $U[v] := M[v] + I$, with the identity matrix I, is a unitary matrix, in the sense that $U[v]^t(\overline{U[v]}) = I$.*

Proof. As the claim is trivial for $[v] = [0]$, we may assume that $[v]$ lies in $\mathbf{PG}(V)$. Then $0 = (m(x,v), m(y,v))$ for any $x, y \in V$ by Proposition 2.2(iv). Putting $m(z,v) = a(z,v) + a(z,v)L[v]$ for $z \in \{x,y\}$, we have

$$0 = (a(x,v), a(y,v)L[v]) + (a(x,v)L[v], a(y,v)) + (a(x,v)L[v], a(y,v)L[v]).$$

Using the matrix $M[v] = (\mu_{ij})$ representing $L[v]$, this equation for x, y with $a(x,v) = \Delta_i$, $a(y,v) = \Delta_{i'}$ reads $0 = (\Delta_i, \sum_{j=0}^{2} \mu_{i'j}\Gamma_j) + (\sum_{j=0}^{2} \mu_{ij}\Gamma_j, \Delta_{i'}) + (\sum_{j=0}^{2} \mu_{ij}\Gamma_j, \sum_{j=0}^{2} \mu_{i'j}\Gamma_j) = \overline{\mu_{i'i}} + \mu_{ii'} + \sum_{j=0}^{2} \mu_{ij}\overline{\mu_{i'j}}$. This is equivalent to the condition that $M[v] + {}^t\overline{M[v]} + M[v]^t\overline{M[v]} = 0$, or equivalently, to the condition that $(M[v] + I)^t\overline{(M[v] + I)} = I$. □

For short, we set $a(x,v) = \sum_{i=0}^{2} y_i \Delta_i$ and $a(x,v)L[v] = \sum_{i=0}^{2} z_i \Gamma_i$. Thus

$$y_i := x_i v_i + \omega(x \times v)_i + \overline{(x \times v)_j (x \times v)_k}, \quad z_i := (x \times v)_i.$$

We shall derive an explicit formula expressing each z_i in terms of y_0, y_1 and y_2. It depends on the shape of a nonzero vector v in V. We separate three cases:

(a) $v_i \neq 0$ but $v_j = v_k = 0$ for $\{i,j,k\} = \{0,1,2\}$.

(b) $v_i = 0$, $v_j \neq 0$ and $v_k \neq 0$ for $\{i,j,k\} = \{0,1,2\}$.

(c) all v_i ($i \in \{0,1,2\}$) are nonzero.

In case (a), we have $z_j = x_k v_i$, $z_k = x_j v_i$, $z_i = 0$; $y_i = x_i v_i + \overline{x_k v_i} \cdot \overline{x_j v_i} = (x_i + \overline{x_j x_k}) v_i$, $y_j = \omega x_k v_i$, $y_k = \omega x_j v_i$. Thus $z_i = 0$, $z_j = \overline{\omega} y_j$, $z_k = \overline{\omega} y_k$. In case (b), we obtain $z_i = x_j v_k + x_k v_j$, $z_j = x_i v_k$, $z_k = x_i v_j$; $y_i = \omega z_i + x_i \overline{v_j v_k}$, $y_j = x_j v_j + \omega x_i v_k + \overline{z_k z_i}$, $y_k = x_k v_k + \omega x_i v_j + \overline{z_j z_i}$. Then the following are verified by straightforward calculations: $z_i = (\overline{v_j} v_k) y_j + (v_j \overline{v_k}) y_k$, $z_j = v_j \overline{v_k} y_i + \omega y_j + \omega \overline{v_j} v_k y_k$, $z_k = v_k \overline{v_j} y_i + \omega y_k + \omega \overline{v_k} v_j y_j$. In case (c), we can verify that $\overline{v_j} y_k + \overline{v_k} y_j = (\pi + \overline{\pi} + \omega) \overline{\pi v_i} z_i$, with $\pi := v_i v_j v_k$ ($\neq 0$) by straightforward calculations using $v_i z_i = v_j z_j + v_k z_k$ for $z = x \times v$. As $\pi + \overline{\pi}$ lies in $\mathbb{F}_2$, $\pi + \overline{\pi} + \omega \neq 0$ and $z_i = \pi(\pi + \overline{\pi} + \overline{\omega}) v_i (\overline{v_k} y_j + \overline{v_j} y_k)$.

Summarizing, we have the following results, where the latter claim immediately follows from Lemma 3.3 and properties (1') and (2') of DHO-sets in Section 3.2 (notice that $U[v] + U[w] = M[v] + M[w]$ for $[v], [w] \in \tilde{V}$):

Lemma 3.4. *The DHO-set $\mathcal{L}_A(\mathcal{M}, Y(g))$ for the DHO $\mathcal{M}$ with respect to $A \in \mathcal{M}$ and the complement $Y(g)$ in Proposition 3.2 consists of the linear maps $L[0] = 0$ and $L[v]$ ($[v] \in \mathbf{PG}(V)$, $v = [v_0, v_1, v_2] = \sum_{i=0}^{2} v_i e_i$) from A to $Y(g)$, represented by the matrices $M[v]$ below with respect to a basis $(\Delta_i)_{i=0}^2$ for A and a basis $(\omega\Delta_i + \nabla_i)_{i=0}^2$ for $Y(g)$, where we set $c := \pi(\pi + \overline{\pi} + \overline{\omega})$, if $\pi := v_0 v_1 v_2 \neq 0$.*

The set $\{U[v] := M[v] + I \mid [v] \in \tilde{V}\}$ consists of 22 unitary matrices of degree 3 over $\mathbb{F}_4$ ($U[v]^t \overline{(U[v])} = I$) such that $\dim(\{x \in V \mid xU[v] = xU[w]\}) = 1$ and $\{x \in V \mid xU[v] = xU[w] = xU[u]\} = \{0\}$ for any mutually distinct $[v]$, $[w]$, $[u]$ in $\tilde{V} = \{0\} \cup \mathbf{PG}(V)$.

$$M[v_0, v_1, v_2] = c \begin{pmatrix} 0 & \overline{v_2} v_0 & v_0 \overline{v_1} \\ v_1 \overline{v_2} & 0 & \overline{v_0} v_1 \\ \overline{v_1} v_2 & v_2 \overline{v_0} & 0 \end{pmatrix}, \quad \textit{if } v_0 v_1 v_2 \neq 0;$$

$$M[0, v_1, v_2] = \begin{pmatrix} 0 & \overline{v_1} v_2 & v_1 \overline{v_2} \\ v_1 \overline{v_2} & \omega & \omega \overline{v_1} v_2 \\ \overline{v_1} v_2 & \omega v_1 \overline{v_2} & \omega \end{pmatrix}, \quad \textit{if } v_1 v_2 \neq 0;$$

$$M[v_0, 0, v_2] = \begin{pmatrix} \omega & \overline{v_2} v_0 & \omega v_2 \overline{v_0} \\ v_2 \overline{v_0} & 0 & \overline{v_2} v_0 \\ \omega \overline{v_2} v_0 & v_2 \overline{v_0} & \omega \end{pmatrix}, \quad \textit{if } v_2 v_0 \neq 0;$$

$$M[v_0, v_1, 0] = \begin{pmatrix} \omega & \omega \overline{v_0} v_1 & v_0 \overline{v_1} \\ \omega v_0 \overline{v_1} & \omega & \overline{v_0} v_1 \\ \overline{v_0} v_1 & v_0 \overline{v_1} & 0 \end{pmatrix}, \quad \textit{if } v_0 v_1 \neq 0;$$

$$
\begin{aligned}
M[v_0,0,0] &= \mathrm{diag}(0,\overline{\omega},\overline{\omega}), && \text{if } v_0 \neq 0;\\
M[0,v_1,0] &= \mathrm{diag}(\overline{\omega},0,\overline{\omega}), && \text{if } v_1 \neq 0;\\
M[0,0,v_2] &= \mathrm{diag}(\overline{\omega},\overline{\omega},0), && \text{if } v_2 \neq 0.
\end{aligned}
$$

3.4 Action of the automorphism group on the complements

In this section, we classify all complements to $\mathcal{M}$. They are 3-dimensional subspaces Y of $S^2(V)$ satisfying $X \cap Y = \{0\}$ for all members $X \in \mathcal{M}$. Notice that the linear automorphism group $L(\mathcal{M})$ acts naturally on the set of complements to $\mathcal{M}$. We shall show the following results.

Proposition 3.5. *There are exactly $2^5.5.7.11$ complements to $\mathcal{M}$, which form a single orbit under the action of $L(\mathcal{M})$. Each of them is a non-degenerate subspace with respect to the restriction $(\ ,\ )$ of the unitary form on $S^2(V)$.*

Proposition 3.5 is obtained by examining orbits under the action of $L(\mathcal{M})$ on the set of d-dimensional subspaces of $S^2(V)$ for $d = 1, 2, 3$. All arguments in the sequel are either geometric or elementary group theoretic, except for Lemma 3.10(2) and (3), where we calculate the stabilizers exploiting explicit matrices obtained from Proposition 2.5 and σ in Lemma 2.6. Following the terminologies in projective geometry, we shall call a d-dimensional subspace *point*, *line* and *plane*, according as $d = 1, 2$ and 3. To distinguish a projective point in $\mathbf{PG}(S^2(V))$ from a projective point in $\mathbf{PG}(V)$, denoted $[v]$ for a nonzero $v \in V$, we use the symbol $\langle u \rangle$ to denote a projective point in $\mathbf{PG}(S^2(V))$ for a nonzero vector u in $S^2(V)$. Recall that $S^2(V)$ is equipped with a unitary form $(\ ,\ )$ preserved by $L(M)$ (see Lemma 2.3) and that every member of $\mathcal{M}$ is totally isotropic with respect to this unitary form. We denote by $\mathcal{P}$ the set of points lying in some members of $\mathcal{M}$. As there are exactly 2 members of $\mathcal{M}$ containing each point in $\mathcal{P}$, we have $|\mathcal{P}| = 22.21/2 = 231$. The group $L(M)$ acts transitively on $\mathcal{P}$, as $L(M)$ is doubly transitive on the members of $\mathcal{M}$.

Take a point $a := \langle \Delta_2 \rangle$ in $\mathcal{P}$. The stabilizer $L(\mathcal{M})_a$ of a in $L(M)$ is the setwise stabilizer of the two members A and $A[e_2]$ of $\mathcal{M}$ containing a, which contains the stabilizer of A and $A[e_2]$ as subgroups of index 2. Thus $L(\mathcal{M})_a = \{\tilde{g} \mid g \in \mathrm{SL}(V), [e_2]g = [e_2]\}\,\langle\sigma\rangle$, where σ (see Lemma 2.6) flips A and $A[e_2]$. Then Proposition 2.5 implies $L(\mathcal{M})_a \cong (3 \times (2^4 : \mathrm{SL}_2(4))) : 2$. As this group preserves the unitary form on $S^2(V)$, $\mathcal{P} \setminus \{a\}$ are naturally divided into the following three classes:

$\mathcal{P}_1(a)$ consisting of points b with $\langle a, b \rangle \subseteq X$ for $X = A$ or $X = A[e_2]\}$,

$\mathcal{P}_2(a)$ consisting of points b orthogonal to a but not lying in $\mathcal{P}_1(a)$,

$\mathcal{P}_3(a)$ consisting of points b which is not orthogonal to a.

Using the triple transitivity of $L(\mathcal{M})$ on the members of $\mathcal{M}$, we can see that $\mathcal{P}_i(a)$ ($i = 0, 1, 2$) are orbits under the stabilizer $L(\mathcal{M})_a$. For $\mathcal{P}_1(a)$, this follows from the doubly transitivity of $\mathrm{SL}(V)$ on the set of projective points in $\mathbf{PG}(V)$ and the existence of the automorphism σ in $L(\mathcal{M})_a$ exchanging two members A and $A[e_2]$ containing a. Now take a point b in $\mathcal{P}$ not lying on A nor $A[e_2]$. By the triple transitivity of $L(\mathcal{M})$ on the members of $\mathcal{M}$, there is a linear automorphism λ of $\mathcal{M}$ stabilizing A and $A[e_2]$ (and hence fixing $a = A \cap A[e_2]$) such that $b\lambda \in A[e_1]$. If b lies in $\mathcal{P}_2(a)$ (resp. $\mathcal{P}_3(a)$), $b\lambda$ is orthogonal (resp. not orthogonal) to a, because λ preserves the unitary form $(\ ,\ )$ by Lemma 2.3. Then, if $b \in \mathcal{P}_2(a)$, $b\lambda$ lies in the line $l := A[e_1] \cap a^{\perp} = \langle \Delta_1, \nabla_0 \rangle$. Notice that the stabilizer of the line l in $L(\mathcal{M})_{A[e_1]} \cong \mathrm{SL}(V)$ (Proposition 2.5) induces A_5 on the five points on l. Thus the stabilizer G of l and two points $\langle \Delta_1 \rangle$ and $\langle \nabla_0 \rangle$ is transitive on the remaining three points, one of which is $b\lambda$. Notice that G stabilizes the unique members A and $A[e_2]$ of $\mathcal{M} \setminus \{A[e_1]\}$ containing $\langle \Delta_1 \rangle$ and $\langle \nabla_0 \rangle$ respectively, whence fixes $a = A \cap A[e_2]$. Thus the transitivity of $L(\mathcal{M})_a$ on $\mathcal{P}_2(a)$ is established. If $b \in \mathcal{P}_3(a)$, then $b\lambda$ is one of the 16 points in $A[e_1]$ outside the line l, on which the point-wise stabilizer K of l in $L(\mathcal{M})_{A[e_1]}$ acts regularly. As K fixes $\langle \Delta_1 \rangle$ and $\langle \nabla_0 \rangle$, K lies in $L(\mathcal{M})_a$. This established the transitivity of $L(\mathcal{M})_a$ on $\mathcal{P}_3(a)$. We have $|\mathcal{P}_2(a)| = 20 \cdot 3/2 = 30$, because each point in $\mathcal{P}_2(a)$ lies in two members in $\mathcal{M} \setminus \{A, A[e_2]\}$ and each member X in $\mathcal{M} \setminus \{A, A[e_2]\}$ contains the line $X \cap a^{\perp}$ consisting of three points in $\mathcal{P}_2(a)$. Furthermore, $|\mathcal{P}_1(a)| = 2 \times 20 = 40$ and $|\mathcal{P}_3(a)| = |\mathcal{P}| - 1 - 40 - 30 = 160$. A point $\langle \Delta_2 + \alpha(\Delta_1 + \nabla_0) \rangle$ ($\alpha \in \mathbb{F}_4$) on the line through $a = \langle \Delta_2 \rangle$ and $b = \langle \Delta_1 + \nabla_0 \rangle$ ($\in \mathcal{P}_2(a)$) lies in $\mathcal{P}$ if and only if $\Delta_2 + \alpha(\Delta_1 + \nabla_0) = m(x, y)$ for some $x, y \in V$. Thus we have $(x \times y)_0 = \alpha$, $(x \times y)_1 = (x \times y)_2 = 0$, and $x_0 y_0 = 0$, $x_1 y_1 = \alpha$ and $x_2 y_2 = 1$. It can be verified that this holds exactly when $\alpha = 1$. Thus $\langle \Delta_2 + \Delta_1 + \nabla_0 \rangle$ is the unique point in $\mathcal{P}$ on the line through a and b other than a and b. If $b = \langle \nabla_2 \rangle \in \mathcal{P}_3(a)$, it is also easy to verify that the third isotropic point $\langle \Delta_2 + \nabla_2 \rangle$ is not in $\mathcal{P}$. Summarizing, we have:

Lemma 3.6. *For each $a \in \mathcal{P}$, there are exactly 4 orbits on $\mathcal{P}$ under the action of the stabilizer $L(\mathcal{M})_a$ of a in $L(\mathcal{M})$. Three obits $\mathcal{P}_i(a)$ ($i = 1, 2, 3$) except the trivial orbit $\{a\}$ are described as follows.*

(i) *$\mathcal{P}_1(a)$ consists of b in $\mathcal{P}$ such that the line through a and b is contained in some members of $\mathcal{M}$. We have $|\mathcal{P}_1(a)| = 40$. For $b \in \mathcal{P}_1(a)$, the isotropic line $\langle a, b \rangle$ consists of 5 isotropic points in $\mathcal{P}$.*

(ii) *$\mathcal{P}_2(a)$ consists of b in $\mathcal{P}$ orthogonal to a (with respect to the unitary form $(\ ,\ )$) but not in $\mathcal{P}_1(a)$. We have $|\mathcal{P}_2(a)| = 30$. For $b \in \mathcal{P}_2(a)$, the isotropic line $\langle a, b \rangle$ contains exactly three isotropic points in $\mathcal{P}$.*

(iii) *$\mathcal{P}_3(a)$ consists of b in $\mathcal{P}$ not orthogonal to a. We have $|\mathcal{P}_3(a)| = 160$. For $b \in \mathcal{P}_3(a)$, the line $\langle a, b \rangle$ contains exactly three isotropic points (which are* *mutually orthogonal) and the unique isotropic point distinct from a and b does not lie in $\mathcal{P}$.*

Now let $\mathcal{L}$ be the set of all isotropic lines in $S^2(V)$. We denote by $a^\perp$ the hyperplane of $S^2(V)$ consisting of vectors orthogonal to an arbitrary isotropic point a. Then $a^\perp/a$ has a nondegenerate unitary form inherited from $(\ ,\ )$. In particular, it has exactly 45 isotropic points which bijectively correspond to the isotropic lines through a. Thus by counting pairs (a, l) of an isotropic point a and an isotropic line l through a, we have $5|\mathcal{L}| = (3^2.7.11).45$, whence $|\mathcal{L}| = 3^4.7.11$. We denote by $\mathcal{L}(i)$ the subset of $\mathcal{L}$ consisting of lines containing exactly i points in $\mathcal{P}$ for $i = 0, \dots, 5$.

Lemma 3.7. *We have $\mathcal{L} = \mathcal{L}(5) \cup \mathcal{L}(3) \cup \mathcal{L}(1)$.*

Proof. For each $i = 1, \dots, 5$, we count the number

$$x_i := |\{(a, l) \mid a \in \mathcal{P}, l \in \mathcal{L}(i), a \in l\}|.$$

We have $x_i = i|\mathcal{L}(i)|$. Fix a point a in $\mathcal{P}$, and let l be an isotropic line through a. Assume that l contains a point b in $\mathcal{P}$ distinct from a. As b is orthogonal to a, it follows from Lemma 3.6 that b lies in $\mathcal{P}_1(a)$ or $\mathcal{P}_2(a)$. If b lies in $\mathcal{P}_1(a)$, the line $l = \langle a, b\rangle$ is one of the $5 + 5$ lines contained in the two members of $\mathcal{M}$ containing a. Thus there are exactly 10 lines in $\mathcal{L}(5)$ through a. If b lies in $\mathcal{P}_2(a)$, the line $l = \langle a, b\rangle$ contains two points in $\mathcal{P}$ distinct from a. Thus there are $30/2 = 15$ lines in $\mathcal{L}(3)$ through a. Accordingly, there are exactly $45 - 10 - 15 = 20$ lines in $\mathcal{L}(1)$ through a, because there are exactly 45 isotropic lines through a as we remarked before stating the lemma. Hence $x_5 = 5|\mathcal{L}(5)| = 10|\mathcal{P}|$, $x_3 = 3|\mathcal{L}(3)| = 15|\mathcal{P}|$ and $x_1 = |\mathcal{L}(1)| = 20|\mathcal{P}|$. Then $|\mathcal{L}(5)| = 2|\mathcal{P}|$, $|\mathcal{L}(3)| = 5|\mathcal{P}|$ and $|\mathcal{L}(1)| = 20|\mathcal{P}|$. As $|\mathcal{L}| = 27|\mathcal{P}|$, this implies that $\mathcal{L}$ is a disjoint union of $\mathcal{L}(5)$, $\mathcal{L}(3)$ and $\mathcal{L}(1)$. □

Lemma 3.8. *Any complement Y to $\mathcal{M}$ is a non-degenerate unitary plane with respect to the restriction of the unitary form $(\ ,\ )$ on Y.*

Proof. Suppose there is a complement Y to $\mathcal{M}$ which is degenerate with respect to the restriction of $(\ ,\ )$. Then the radical R of Y (the subspace of Y consisting of vectors orthogonal to all vectors in Y) is of dimension $1, 2$ or 3. In the latter two cases, R contains an isotropic line, and therefore R contains a point in $\mathcal{P}$ by Lemma 3.7, which contradicts that Y is a complement to $\mathcal{M}$. Thus R is of dimension 1. However, in this case, there is a line l of Y with $Y = l \oplus R$. As any line of $S^2(V)$ contains at least one isotropic point, l contains an isotropic point p and then the isotropic line $\langle p, R\rangle$ contains a point in $\mathcal{P}$, which is again a contradiction. □

It follows from Lemma 3.8 above that any complement is non-degenerate with respect to the restriction of the unitary form $(\ ,\ )$. We denote by $\mathcal{S}$ the set of non-degenerate planes in $S^2(V)$. For a non-negative integer i, we denote by $\mathcal{S}(i)$ the set of non-degenerate planes which contains exactly i points in $\mathcal{P}$. Thus $\mathcal{S}(0)$ coincides with the set of complements to $\mathcal{M}$.

A non-degenerate plane Y of V has a structure of a 3-dimensional unitary space over $\mathbb{F}_4$. Thus it is spanned by three anisotropic points n_i $(i = 1, 2, 3)$ which are mutually orthogonal. Observe that each line $l_{ij} := \langle n_i, n_j \rangle$ $(1 \leq i < j \leq 3)$ contains exactly three isotropic points. A line spanned by n_i and an isotropic point a with $(a, n_i) = 0$ $(i = 1, 2, 3)$ contains 4 anisotropic points. Thus a nondegenerate plane Y contains exactly 9 isotropic and 12 anisotropic points. In particular, the 9 isotropic points on Y are divided into three subsets on l_{ij} $(1 \leq i < j \leq 3)$. As there is no isotropic line in Y, any two distinct isotropic points are non-orthogonal.

The number $|\mathcal{S}|$ is obtained by counting the quadruple (n_1, n_2, n_3, Y) of mutually orthogonal anisotropic points n_i $(i = 1, 2, 3)$ of V and $Y \in \mathcal{S}$ containing all n_i $(i = 1, 2, 3)$. There are exactly $((4^6 - 1)/(4 - 1)) - 693 = 672$ anisotropic points in $S^2(V)$. If we fix one of them, say n_1, the subspace $n_1^{\perp}$ orthogonal to n_1 is a 5-subspace of $S^2(V)$ with a nondegenerate unitary form inherited from $(\ ,\ )$. It is known that there are exactly $((4^5 - 1)/(4 - 1)) - 165 = 176$ anisotropic points in $n_1^{\perp}$. Taking one of them, say n_2, we have $((4^4 - 1)/(4 - 1)) - 45 = 40$ anisotropic points in $\langle n_1, n_2 \rangle^{\perp}$, which is a 4-subspace of $S^2(V)$ with a nondegenerate unitary form inherited from $(\ ,\ )$. As $Y = \langle n_1, n_2, n_3 \rangle$, there are exactly $672 \cdot 176 \cdot 40 = 2^{12}.3.5.7$ quadruples. On the other hand, if we fix $Y \in \mathcal{S}$, the number of triples (n_1, n_2, n_3) of mutually orthogonal anisotropic points contained in Y is given by $12 \cdot 2 \cdot 1 = 2^3.3$ from the description in the above paragraph. Hence we obtain $|\mathcal{S}| = 2^9.5.7.11$.

Lemma 3.9. (1) *Every non-degenerate plane has at most* 4 *points in* $\mathcal{P}$.

(2) *We have* $|\mathcal{S}(4)| = |\mathcal{S}(2)| + 2|\mathcal{S}(1)| + 3|\mathcal{S}(0)|$.

Proof. (1) Suppose that a non-degenerate plane Y of $S^2(V)$ contains five points in $\mathcal{P}$. Take mutually orthogonal anisotropic points n_i $(i = 1, 2, 3)$. We may assume that the lines $l_{12} := \langle n_1, n_2 \rangle$ and $l_{13} := \langle n_1, n_3 \rangle$ respectively contain two isotropic points a_1, a_2 and b_1, b_2 in $\mathcal{P}$. As a_1 and a_2 are non-orthogonal, the third isotropic point a_3 on l_{12} is not contained in $\mathcal{P}$ in view of Lemma 3.6(iii). Similarly, the the third isotropic point b_3 on l_{13} is not contained in $\mathcal{P}$. Now the line through a_1 and b_i $(i = 1, 2)$ intersects the line $l_{23} := \langle n_2, n_3 \rangle$ at a point, say c_i. As $b_i \notin \langle a_1, n_2 \rangle = l_{12}$ and $a_1 \notin \langle b_i, n_3 \rangle = l_{13}$, $c_i \notin \{n_2, n_3\}$. In particular, $c_i$is isotropic. But $c_i \notin \mathcal{P}$ by Lemma 3.6(iii) applied to non-orthogonal points a_1 and b_i in $\mathcal{P}$. Then the remaining isotropic points, say c_3 on l_{23} must be in $\mathcal{P}$ by our assumption. Now c_3 and a_i $(i = 1, 2)$ are non-orthogonal points in $\mathcal{P}$ so that the intersection of lines $\langle a_i, c_3\rangle$ and l_{13} should be an isotropic point not in $\mathcal{P}$. Thus these lines should intersect at b_3. However, this implies that $\langle a_1, c_3\rangle = \langle b_3, c_3\rangle = \langle a_2, c_3\rangle$, which is impossible.

(2) We count the number N of pairs (a, Y) of points a of $\mathcal{P}$ and a plane Y in $\mathcal{S}$ with $a \in Y$. Fixing $Y \in \mathcal{S} = \cup_{i=0}^{4}\mathcal{S}(i)$ first, we have $N = \sum_{i=1}^{4} i|\mathcal{S}(i)|$. On the other hand, fix a point a of $\mathcal{P}$. Assume that Y is a nondegenerate plane of $S^2(V)$ through a. Then Y/a is a line in the 5-dimensional space $S^2(V)/a$. The non-degeneracy of Y implies that Y/a is not contained in the 4-subspace $a^{\perp}/a$, and that $(Y \cap a^{\perp})/a$ is a anisotropic point in $a^{\perp}/a$ (with respect to the nondegenerate unitary form inherited from (,)). Conversely, for every line of $S^2(V)/a$ with these properties, its inverse image in $S^2(V)$ is a nondegenerate plane of $S^2(V)$ through a. Now, the number of such lines, or equivalently the number of $Y \in \mathcal{S}$ through a is counted as follows. $S^2(V)/a - (a^{\perp}/a)$ contains 4^4 points x and $a^{\perp}/a$ contains 40 anisotropic points y. As $Y/a = \langle x, y\rangle$ contains precisely 4 points, there are exactly $40 \cdot (4^4/4) = 2^9.5$ distinct $Y \in \mathcal{S}$ with $a \in Y$. Thus $N = |\mathcal{P}| \cdot 2^9.5$. As $N = 2^9.3.5.7.11 = 3|\mathcal{S}|$ and $|\mathcal{S}| = \sum_{i=0}^{4} |\mathcal{S}(i)|$, the formula in the statement of the lemma follows from $N = \sum_{i=1}^{4} i|\mathcal{S}(i)|$. □

We now consider the following two non-degenerate planes Y_0 and Y_2.

$$Y_0 := \langle \Delta_i + \overline{\omega}\nabla_i \mid i \in \{0,1,2\}\rangle\,,$$
$$Y_2 := \langle \Delta_0 + \omega\nabla_0, \Delta_0 + \overline{\omega}\nabla_0, \Delta_1 + \overline{\omega}\nabla_1\rangle\,.$$

We shall calculate the stabilizer of Y_i in $L(\mathcal{M})$ for $i = 0, 1$. Here we exploit some explicit linear bijections obtained by Proposition 2.5.

Lemma 3.10. (1) *$Y_0 \in \mathcal{S}(0)$ and $Y_2 \in \mathcal{S}(2)$.*

(2) *The stabilizer of Y_0 in $L(\mathcal{M})$ is of order $2^2.3^3$ (isomorphic to $3_+^{1+2} : Z_4$).*

(3) *The stabilizer of Y_2 in $L(\mathcal{M})$ is of order 2.3^2 (isomorphic to $E_{3^2} \times 2$).*

Proof. (1) Notice that Y_0 coincides with the plane $Y(g)$ in Proposition 3.2. Thus $Y_0 \in \mathcal{S}(0)$. The plane Y_2 is a nondegenerate unitary plane spanned by three mutually orthogonal anisotropic points $a_0 := \langle \Delta_0 + \omega\nabla_0\rangle$, $a_1 := \langle \Delta_0 + \overline{\omega}\nabla_0\rangle$ and $a_2 := \langle \Delta_1 + \overline{\omega}\nabla_1\rangle$. Thus any isotropic point in Y_2 lies in a line l_{ij} through a_i and a_j for some $0 \le i < j \le 2$. As l_{12} is a subset of Y_0, it does not contain a point in $\mathcal{P}$. Observe that l_{01} contains two points $\langle \Delta_0\rangle$ and $\langle \nabla_0\rangle$ in $\mathcal{P}$, which are non-orthogonal. Thus they are the only points in $\mathcal{P}$ lying on l_{01} by Lemma 3.6(iii). Take an isotropic point $\langle a_0 + \alpha a_2\rangle$

(α is a nonzero element in $\mathbb{F}_4$) on the line l_{02}. Suppose $m(x,y) = a_0 + \alpha a_2$ ($= \Delta_0 + \alpha\Delta_1 + \omega\nabla_0 + \alpha\overline{\omega}\nabla_1$) for some $x, y \in V$. Comparing the coefficients of the basis vectors, we have $(x \times y)_0 = \omega$, $(x \times y)_1 = \alpha\overline{\omega}$, $(x \times y)_2 = 0$, $x_0y_0 = 1$, $x_1y_1 = \alpha$ and $x_2y_2 = \overline{\alpha}$. Then $y_2 = \overline{\alpha x_2}$ and $y_1 = \alpha\overline{x_1}$, and therefore $\omega = (x \times y)_0 = x_1y_2 + x_2y_1 = \overline{\alpha}x_1\overline{x_2} + \alpha\overline{x_1}x_2$. However, as the last term is an element of $\mathbb{F}_2$, this is a contradiction. Thus there is no points in $\mathcal{P}$ on the line l_{02}, and Y_2 contains just two points $\langle\Delta_0\rangle$ and $\langle\nabla_0\rangle$ in $\mathcal{P}$.

(2) Recall that the linear bijections on a non-degenerate unitary plane Y preserving the unitary form form a group denoted $\mathrm{GU}_3(2)$, which is isomorphic to $3 \cdot (E_9 : Q_8)$, where Q_8 denotes the quaternion group of order 8 and E_9 denotes the elementary abelian group of order 9. In particular, this group has a largest normal 3-subgroup isomorphic to 3^{1+2}_+, the extraspecial group of order 3^3. Thus the stabilizer of Y_0 in $L(\mathcal{M})$ induces a subgroup of $\mathrm{GU}_3(2)$. A subgroup corresponding to 3^{1+2}_+ is constructed as follows: take linear maps $t_1 := (e_0, e_1, e_2)$ and $t_2 := \mathrm{diag}(1, \omega, \overline{\omega})$ in $\mathrm{SL}(V)$, where $\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2)$ denotes the diagonal matrix sending e_i to $\alpha_i e_i$ $(i = 0, 1, 2)$ and (e_0, e_1, e_2) denotes the permutation matrix sending e_i to e_{i+1}, reading suffixes modulo 3. It is straightforward to verify that t_1 and t_2 generate a subgroup of $\mathrm{SL}(V)$ isomorphic to 3^{1+2}_+ with center generated by ωI_3. From Proposition 2.5, we can obtain the following linear maps $\tilde{t}_1$ and $\tilde{t}_2$ in $L(\mathcal{M})$ (stabilizing A): $\tilde{t}_1 = (\Delta_0, \Delta_1, \Delta_2)(\nabla_0, \nabla_1, \nabla_2)$, $\tilde{t}_2 = \mathrm{diag}(1, \overline{\omega}, \omega, 1, \overline{\omega}, \omega)$, where we use the above convention to denote monomial matrices with respect to the (ordered) basis $(\Delta_0, \Delta_1, \Delta_2, \nabla_0, \nabla_1, \nabla_2)$ for $S^2(V)$. Then $\tilde{t}_1$ and $\tilde{t}_2$ generate a subgroup $\tilde{T}$ of $L(\mathcal{M})$ isomorphic to 3^{1+2}_+ (with center $\langle\omega I_6\rangle$ (corresponding to $\langle\omega I_3\rangle$). Furthermore, they acts on the basis vectors $a_i := \Delta_i + \overline{\omega}\nabla_i$ $(i = 0, 1, 2)$ for Y_0 in the following manner: $a_i\tilde{t}_1 = a_{i+1}$ (reading the suffixes modulo 3) and $a_i\tilde{t}_2 = \omega^{-i}a_i$ $(i = 0, 1, 2)$. Thus $\tilde{T}$ is a subgroup of the stabilizer $L(\mathcal{M})_{Y_0}$ which corresponds to the largest normal 3-subgroup of $\mathrm{GU}_3(2)$.

This also implies that the stabilizer $L(\mathcal{M})_{Y_0}$ is contained in the normalizer of $\tilde{T}$ in $L(\mathcal{M})$, as $|L(\mathcal{M})|_3 = 3^2$. As $L(\mathcal{M})/\langle\omega I_6\rangle$ is a simple group of order $2^7.3^2.5.7.11$, the factor group $\tilde{T}/\langle\omega I_6\rangle$ is a Sylow 3-subgroup and it follows from the Sylow theorem that the normalizer in the factor group has order $2^3.3^2$. We can find a Sylow 2-subgroup of the normalizer of $\tilde{T}$ in $L(\mathcal{M})$ as follows: take linear maps q_1 and q_2 of $\mathrm{SL}(V)$ represented by the matrices

$$q_1 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \omega \\ 1 & \omega & \overline{\omega} \end{pmatrix} \quad \text{and} \quad q_2 := \begin{pmatrix} 1 & \overline{\omega} & \overline{\omega} \\ \omega & \omega & \overline{\omega} \\ \omega & \overline{\omega} & \omega \end{pmatrix}.$$

It is easy to see that $(q_1)^2 = (q_2)^2 = (e_1, e_2)$ and $q_1^{q_2} = q_1^{-1}$, and hence

$\langle q_1, q_2\rangle \cong Q_8$. Furthermore, we can verify that $t_1^{q_1} = t_2$, $t_2^{q_1} = t_1$, $t_1^{q_2} = t_1 t_2(\omega I_3)^2$ and $t_2^{q_2} = t_1 t_2^{-1}(\omega I_3)$. Thus $\langle q_1, q_2\rangle$ normalizes T. From Proposition 2.5, we can obtain the following linear maps $\tilde{q}_1$ and $\tilde{q}_2$ in $L(\mathcal{M})$ (stabilizing A) with some calculations:

$$\tilde{q}_1 := \begin{pmatrix} \overline{q_1} & 0 \\ 0 & \overline{q_1} \end{pmatrix} \quad \text{and} \quad \tilde{q}_2 := \begin{pmatrix} \overline{q_2} & 0 \\ \overline{q_2} & \overline{q_2} \end{pmatrix},$$

where $\overline{A} := (\overline{a_{ij}})$ for $A = (a_{ij})$. Thus $\tilde{Q} := \langle \tilde{q}_1, \tilde{q}_2\rangle$ ($\cong Q_8$) is a Sylow 2-subgroup of the normalizer of $\tilde{T}$ in $L(\mathcal{M})$.

As we remarked above, the stabilizer of the plane Y_0 in $L(\mathcal{M})$ is a subgroup of the normalizer of $\tilde{T}$ (which is $\tilde{T}\tilde{Q}$) containing $\tilde{Q}$. It is straightforward to see that that $\tilde{q}_1$ preserves Y_0 but $\tilde{q}_2$ does not. In fact, we have $a_0\tilde{q}_1 = a_0 + a_1 + a_2$, $a_1\tilde{q}_1 = a_0 + \overline{\omega} a_1 + \omega a_2$ and $a_2\tilde{q}_1 = a_0 + \omega a_1 + \overline{\omega} a_2$; but $a_1\tilde{q}_2 = (\Delta_0 + \omega\nabla_0) + (\Delta_1 + \omega\nabla_1) + \overline{\omega}(\Delta_2 + \omega\nabla_2)$, which is not contained in Y_0. (In fact, for the semi-linear automorphism ι on $S^2(V)$ sending a vector $\sum_{i=0}^2 (x_i\Delta_i + y_i\nabla_i)$ to $\sum_{i=0}^2 (\overline{x_i}\Delta_i + \overline{y_i}\nabla_i)$, we can verify that ι is an automorphism of $\mathcal{M}$ and that $\tilde{q}_2\iota$ preserves Y_0.) As $\tilde{T}\langle \tilde{q}_1\rangle$ is a maximal subgroup of the normalizer $\tilde{T}\tilde{Q}$, we conclude that the stabilizer of Y_0 in the linear automorphism group $L(\mathcal{M})$ coincides with $\tilde{T}\langle \tilde{q}_1\rangle$ ($\cong 3_+^{1+2} : Z_4$).

(3) As $a := \langle \Delta_0\rangle$ and $b := \langle \nabla_0\rangle$ are the only points of $\mathcal{P}$ contained in Y_2 by claim (1), the stabilizer $L(\mathcal{M})_{Y_2}$ of the plane Y_2 in $L(\mathcal{M})$ is a subgroup of the setwise stabilizer of $\{a, b\}$ in $L(\mathcal{M})$. As a and b are not orthogonal, b lies in the suborbit $\mathcal{P}_3(a)$ in Lemma 3.6. Then the poitwise stabilizer $L(\mathcal{M})_{a,b}$ is a subgroup of $L(\mathcal{M})_a$ ($\cong 3.2^4A_5.2$) of index $|\mathcal{P}_3(a)| = 160$, so that $|L(\mathcal{M})_{a,b}| = 2^2.3^2$. As the automorphism σ in Lemma 2.6 fixes Δ_0 and ∇_0 but exchanges Δ_2 and ∇_2, we conclude that the set wise stabilizer of $\{a, b\}$ in $L(\mathcal{M})$ coincides with the group $\left\langle \omega I_6, \tilde{t}_2, \tilde{q}_1^2, \tilde{t}_1\sigma\tilde{t}_1^{-1}\right\rangle \langle\sigma\rangle$, where $\tilde{t}_i$ ($i = 1, 2$) are the automorphisms of $\mathcal{M}$ appeared in the proof of claim (2). It is straightforward to determine the stabilizer of Y_2 in $L(\mathcal{M})$ coincides with $\left\langle \omega I_6, \tilde{t}_2, \sigma\right\rangle$ ($\cong E_9 \times 2$), working inside this small group. □

Finally the following lemma establishes Proposition 3.5, because the set of complements coincides with $\mathcal{S}(0)$ by Lemma 3.8.

Lemma 3.11. *We have $\mathcal{S} = \mathcal{S}(0) \cup \mathcal{S}(2) \cup \mathcal{S}(4)$, where $\mathcal{S}(i)$ coincides with the orbit under $L(\mathcal{M})$ containing Y_i for every $i \in \{0, 2\}$.*

Proof. By Lemma 3.10, we have $|\mathcal{S}(0)| \geq |L(\mathcal{M})|/2^2.3^3 = 2^5.5.7.11$ and $|\mathcal{S}(2)| \geq |L(\mathcal{M})|/2.3^2 = 2^6.3.5.7.11$. From Lemma 3.9(ii), $|\mathcal{S}(4)| = |\mathcal{S}(2)| + 2|\mathcal{S}(1)| + 3|\mathcal{S}(0)| \geq (2^6.3 + 2^5.3)\cdot t$, where $t := 5.7.11$. Then we have $|\mathcal{S}| \geq |\mathcal{S}(0)| + |\mathcal{S}(2)| + |\mathcal{S}(4)| \geq 2^5(1+6+9)t = 2^9 t$. As we obtained $|\mathcal{S}| = 2^9 t$ in the paragraph previous to Lemma 3.9, this implies that $\mathcal{S} = \mathcal{S}(0) \cup \mathcal{S}(2) \cup \mathcal{S}(4)$ and each $\mathcal{S}(i)$ ($i = 0, 2$) is a single orbit under $L(\mathcal{M})$. $\square$

3.5 Comments about complements

In Section 3.4, we establish the transitivity of $L(\mathcal{M})$ on the set of complements by examining the action of $L(\mathcal{M})$ on various subspaces. As we saw above, this requires somewhat lengthy arguments. Other possible approaches to establish the transitivity will be discussed below.

First, the transitivity would be established in a much shorter and direct way (just by quoting the Sylow theorem), if we can associate a Sylow 3-subgroup of $L(\mathcal{M})$ with each complement $Y(g)$ (see Lemma 3.1). Notice that Proposition 3.5 implies that there are exactly $2^5.5.7.11$ maps g from $V \times V$ to V satisfying conditions (i) and (ii) in Lemma 3.1, because for such maps g and h we have $Y(g) = Y(h)$ if and only if $g = h$.

Next, the transitivity would be established by directly observing the action of $L(\mathcal{M})$ on the set of maps h satisfying conditions (i) and (ii) in Lemma 3.1. First, notice that the alternating bilinear map h is uniquely determined by the matrix $H = (h_{ai})$ of degree 3 with $\sum_{i=0}^{2} h_{ai} e_i = h(e_j, e_k)$ for each $a \in \{0, 1, 2\} = \{i, j, k\}$. Take an automorphism $\tilde{g}$ for $g \in \mathrm{SL}(V)$ (see Proposition 2.5). It is not difficult to see that the subspace $Y(h)\tilde{h} = Y(h')$ corresponds to the matrix $H' := {}^t\overline{G} H G + L(\overline{G})\overline{G}$, where $L(\overline{G})$ is defined at Section 2.3.2.

4 SubDHOs, covers and quotients

In this section, we collect informations of the substructures, covering and quotients of a DHO $\mathcal{M}$ constructed in Section 2.2. Recall that a DHO $\mathcal{S}$ of rank l ($l \geq 2$) over $\mathbb{F}_4$ is called a *subDHO* of $\mathcal{M}$, if any member of $\mathcal{S}$ is a subspace of a member of $\mathcal{M}$. As $\mathcal{M}$ is of rank 3, any proper subDHO has rank 2. Recall also that a DHO $\mathcal{S}$ of rank 3 over $\mathbb{F}_4$ is called a *cover* (resp. *quotient*) of $\mathcal{M}$ if there is a surjective semilinear map ρ from the ambient space of $\mathcal{S}$ to U, the ambient space of $\mathcal{M}$, which sends each member of $\mathcal{S}$ to a member of $\mathcal{M}$ (resp. if there is a surjective semilinear map ρ' from U to the ambient space of $\mathcal{S}$ which sends each member of $\mathcal{M}$ to a member of $\mathcal{S}$).

Let $\mathcal{S}$ be a proper subDHO of $\mathcal{M}$. Since $L(\mathcal{M})$ is transitive on the members of $\mathcal{M}$ and the stabilizer of A in $L(\mathcal{M})$ is transitive on the hyperplanes of A, there is an automorphism α of $\mathcal{M}$ such that $\mathcal{S}^\alpha$ contains $\Delta(W) := \{\Delta(x) \mid x \in W\}$ as a member, where W is a hyperplane of V spanned by e_0 and e_1. For each projective point $[v]$ in $\mathbf{PG}(W)$, we denote by $S[v]$ the unique member of $\mathcal{S}$

distinct from $\Delta(W)$ containing $[v]$. Then $S[v]$ is a hyperplane of $A[v]$. As $\mathcal{S}$ is a DHO, $S[v] \cap S[w]$ is a 1-dimensional subspace for distinct projective points $[v]$ and $[w]$ in $\mathbf{PG}(B)$. On the other hand, $A[v] \cap A[w]$ is a 1-dimensional subspace of U spanned by $m(v,w)$. Thus we have $S[v] \cap S[w] = A[v] \cap A[w]$. Notice that $v \times w = (v_0 w_1 + w_0 v_1)e_2$ for $v, w \in W$, so that $m(v,w) = v \otimes w + \Delta(\iota(v \times w)) = v \otimes w$. As a 2-dimensional subspace $S[v]$ is spanned by $\Delta(v)$ and $m(v,w) = v \otimes w$, we conclude that $S[v] = \{v \otimes x \mid x \in W\}$ for each $0 \neq v \in W$. Thus $\mathcal{S}^{\alpha} = \{\Delta(W)\} \cup \{S[v] \mid [v] \in \mathbf{PG}(W)\}$ coincides with the Veronesean DHO $\mathcal{V}_2(\mathbb{F}_4)$ with ambient space $S^2(W)$.

As the ambient space of any DHO of rank 3 over $\mathbb{F}_4$ has dimension at most 6 by, e.g., [8, Theorem 1(i)], any cover of $\mathcal{M}$ is isomorphic to $\mathcal{M}$. Finally, there is no proper quotient of $\mathcal{M}$, by the following observation to any DHO of polar type, applied to the unitary DHO $\mathcal{M}$.

Lemma 4.1. *Let $\mathcal{S}$ be a DHO of rank n over $\mathbb{F}_q$ with ambient space A of dimension $2n$ equipped with a non-degenerate alternating, quadratic or hermitian form f, in which any member of $\mathcal{S}$ is totally isotropic with respect to f. Then there is no proper quotient of $\mathcal{S}$.*

Proof. Suppose there is a proper quotient $\overline{\mathcal{S}}$ of $\mathcal{S}$. Let π be a covering map from $\mathcal{S}$ to $\overline{\mathcal{S}}$. As the ambient space of a DHO of rank n is at least $2n-1$, the kernel K of a semilinear map π is of dimension 1. Let k be a nonzero vector of K. Recall that $K \cap (X+Y) = \{0\}$ for any distinct member X and Y of $\mathcal{S}$ by [8, Proposition 13].

On the other hand, as f is non-degenerate, $k^{\perp} := \{v \in A \mid f(v,k) = 0\}$ is a hyperplane of A. As $n > 1$, the intersection $k^{\perp} \cap X$ for any member X of $\mathcal{S}$ is of dimension at least 1. Thus there is a nonzero vector x of X such that $f(x,k) = 0$. Let Y be the unique member of $\mathcal{S} \setminus \{X\}$ containing x. Then $X + Y = x^{\perp}$, as $X + Y$ is a hyperplane of A containing isotropic subspaces X and Y. As $k \in x^{\perp}$, this contradicts the fact that $K \cap (X+Y) = \{0\}$. □

Summarizing, we have

Proposition 4.2. *Any proper subDHO of $\mathcal{M}$ is conjugate under the action of $\mathrm{Aut}(\mathcal{M})$ to the Veronesean DHO $\mathcal{V}_2(\mathbb{F}_4)$ with ambient space $S^2(W)$ of dimension 3, where W is a hyperplane of V spanned by e_0 and e_1. There is no proper cover nor proper quotient of $\mathcal{M}$.*

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